

A Study of the Impact of a *Bonus-Malus* System in Finite and Continuous Time Ruin Probabilities in Motor Insurance

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This work was partially supported by the Fundação para a Ciência e a Tecnologia through the Programme through the programme UID/MAT/00297/2013 (CMA) and Programme PEst-OE/EGE/UI0491/2013 (CEMAPRE)

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Goal

- Evaluate (numerically) the ruin probability in continuous and finite time,
- The premium is received at a constant rate during each year,
 - P_0 - Constant for $0 \leq t \leq n$,
 - Changing every year - Bonus Malus System (BMS).
- Bonus Malus System,
 - Classical model,
 - Open model.
- Model for large portfolios,
- Use calculation and simulation,
 - Simulation of the annual aggregate claims amounts,
 - Estimation of the premiums amounts received each year,
 - Calculation of the end of period (year) surplus,
 - Calculation of the ruin probability during the period.

The surplus process

$U(t)$ is the insurer's surplus at time t , $0 \leq t \leq n$

$$U(t) = u + \sum_{j=1}^{i-1} P_j + (t - i + 1)P_i - S(t),$$
$$i = 1, \dots, n$$

i is such that $t \in [i - 1, i)$ and $\sum_{j=1}^0 P_j = 0$

- u is the initial surplus
- P_i is the premium charged in year i (function of the past number of claims)
- $S(t)$ is the aggregate claims up to time t

Probability of ruin within the year $\psi(u(i-1), 1, u(i))$ is the probability of ruin within the year i , given the surplus $u(i-1)$ at the start of the year and the surplus $u(i) = y$ at the end of the year and the rate of premium income p_i during the year.

Following Dickson and Waters (2006).

Method: Dickson and Waters (2006)

$$\psi(u(i-1), 1, u(i)) =$$

$$\frac{\int_{s=0}^{1-u(i)/p_i} f(u(i-1) + p_i s, s) \frac{u(i)}{(1-s)} f(p_i(1-s) - u(i), 1-s) ds}{f(u(i-1) + p_i - u(i), 1)}$$

$$+ \frac{f(u(i-1) + p_i - u(i), 1 - u(i)/p_i) \exp(-\lambda u(i)/p_i)}{f(u(i-1) + p_i - u(i), 1)}$$

- Exact formula to calculate $\psi(u(i-1), 1, u(i))$;
- We need values for $f(\cdot, s)$ to s from 0 to 1;
- Time consuming task for high λ .



We need a numerical method to approximate $\psi(u(i-1), 1, u(i))$.

Method: Dickson and Waters (2006)

- We are going to approximate $f_S(\cdot, s)$ by a translated gamma distribution chosen to match the first three moments.
- α , β and κ are such that:

$$\frac{2}{\sqrt{\alpha}} = \frac{E[(Y_i - E[Y_i])^3]}{\text{Var}[Y_i]^{3/2}}$$

$$\frac{\alpha}{\beta^2} = \text{Var}[Y_i]$$

$$\frac{\alpha}{\beta} + \kappa = E[Y_i]$$

- $H(s) \sim G(\cdot; \alpha s, \beta)$
- For $0 < s \leq 1$ the random variable $H(s) + \kappa s$ has a translated gamma distribution and the first three moments are the same as $S(s)$.

In practice:

$$\begin{array}{ll} f(x, s) & \text{replaced by } f_G(x - \kappa s, s) \\ \exp(-\lambda t) & \text{replaced by } F_G(-\kappa t, t) \end{array}$$

$$\psi(u(i-1), 1, u(i)) \sim \psi_{TG}(u(i-1), 1, u(i))$$

$$\psi_{TG}(u(i-1), 1, u(i)) =$$

$$\frac{\int_{s=0}^{1-u(i)/p_i} f_G(u(i-1) + (p_i - \kappa)s, s) \frac{u(i)}{(1-s)} f_G((p_i - \kappa)(1-s) - u(i), 1-s) ds}{f_G(u(i-1) + p_i - \kappa - u(i), 1)}$$

$$+ \frac{f_G(u(i-1) + (p_i - \kappa)(1 - \frac{u(i)}{p_i}), 1 - \frac{u(i)}{p_i}) F_G(-\kappa u(i)/p_i, u(i)/p_i)}{f_G(u(i-1) + p_i - \kappa - u(i), 1)}$$

In practice:

- Approximation based on three moments;
- There are fast algorithms for gamma distributions;
- Fast results.

Bonus Malus Systems

- BMS are a posterior rating systems correcting the premiums according to past claim experience,
- Reduce the premium for insureds with low claim frequency rate and raise it for insureds with high claim frequencies,
- BMS can be modeled using Markov Chains,
- BMS are characterized by set of Transition Rules, Initial Class and Premium Scale.

What is the impact of BMS or/and Premium Scale in the Probability of Ruin?

Procedure: Afonso *et al.* (2009)

- starting at $u = U(0)$;
- simulate the aggregate claims, $\{y_i\}_{i=1}^n$,
(using the translated gamma approximation, n. of claims in class j is Poisson distributed with parameter λ_j)
- calculate p_i (p_1 known), (Expected value principle, Commercial Scale, 6 Optimal Scales)
- calculate $u(i) = u(i-1) + p_i - y_i$;
- calculate $\psi_{TG}(u(i-1), 1, u(i))$;
 - If $u(i) < 0$ for any $i, i = 1, \dots, n$ then ruin has occurred $\rightarrow \psi_j(u, n) = 1$ and we start another run $j+1$.
 - If $u(i) \geq 0$ for all $i, i = 1, \dots, n$ calculate $\psi(u(i-1), 1, u(i))\dots$

$$\bullet \psi_j(u, n) = 1 - \prod_{i=1}^n \left(1 - \psi_{TG}(u(i-1), 1, u(i))\right).$$

Mean of the N runs \rightarrow final estimate of $\psi(u, n)$.

Example: Portuguese Insurer (2010)

Claims amount	Y	$E(Y)$	1,766
		$V(Y)$	71,097,953
		$\gamma(Y)$	35.1
		$\gamma_2(Y)$	1,816

No. of claims $N \sim \text{Poisson}(\Lambda)$,

$\Lambda \sim \text{InverseGaussian}(\mu = 0.082401, \eta = 0.130271)$

Total policies 442,490,

Time horizon $n = 10$,

No. runs 50,000

Example - One step transition matrix for the BMS

$$T = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \\ 15 \\ 16 \\ 17 \\ 18 \end{matrix} & \left[\begin{array}{cccccccccccccccc} p_0 & - & - & p_1 & - & - & p_2 & - & - & p_3 & - & - & p_4 & - & - & p_5 & - & 1 - \sum_{i=0}^5 p_i \\ p_0 & - & - & - & p_1 & - & - & p_2 & - & - & p_3 & - & - & p_4 & - & - & p_5 & - & 1 - \sum_{i=0}^5 p_i \\ - & p_0 & - & - & - & p_1 & - & - & p_2 & - & - & p_3 & - & - & p_4 & - & - & 1 - \sum_{i=0}^4 p_i \\ - & - & p_0 & - & - & - & p_1 & - & - & p_2 & - & - & p_3 & - & - & p_4 & - & 1 - \sum_{i=0}^4 p_i \\ - & - & - & p_0 & - & - & - & p_1 & - & - & p_2 & - & - & p_3 & - & - & p_4 & 1 - \sum_{i=0}^3 p_i \\ - & - & - & - & p_0 & - & - & - & p_1 & - & - & p_2 & - & - & p_3 & - & - & 1 - \sum_{i=0}^3 p_i \\ - & - & - & - & - & p_0 & - & - & - & p_1 & - & - & p_2 & - & - & p_3 & - & 1 - \sum_{i=0}^2 p_i \\ - & - & - & - & - & - & - & p_0 & - & - & - & p_1 & - & - & p_2 & - & - & 1 - \sum_{i=0}^2 p_i \\ - & - & - & - & - & - & - & - & p_0 & - & - & - & p_1 & - & - & p_2 & - & 1 - \sum_{i=0}^1 p_i \\ - & - & - & - & - & - & - & - & - & p_0 & - & - & - & p_1 & - & - & - & 1 - \sum_{i=0}^1 p_i \\ - & - & - & - & - & - & - & - & - & - & p_0 & - & - & - & - & p_1 & - & 1 - \sum_{i=0}^0 p_i \\ - & - & - & - & - & - & - & - & - & - & - & p_0 & - & - & - & - & - & 1 - p_0 \\ - & - & - & - & - & - & - & - & - & - & - & - & - & p_0 & - & - & - & 1 - p_0 \\ - & - & - & - & - & - & - & - & - & - & - & - & - & - & p_0 & - & - & 1 - p_0 \\ - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & p_0 & - & 1 - p_0 \end{array} \right] \end{matrix}$$

Classical Model for SBM

Premiums $P_0 = P = (1 + \theta)E[Y]E[N_i], i \in \{1, \dots, n\},$

$C = P \times$ Commercial Scale,

$N = P \times$ Norberg Scale,

$NL = P \times$ Gilde and Sund (1989) applied to $N,$

$NG = P \times$ Andrade e Silva & Centeno(2005) applied to $N,$

$B = P \times$ BorganScale,

$BL = P \times$ Gilde and Sund (1989) applied to $B,$

$BG = P \times$ Andrade e Silva & Centeno(2005) applied to $B.$

Initial Surplus varying with the Premium type

Safety loading $\theta = 0.8$

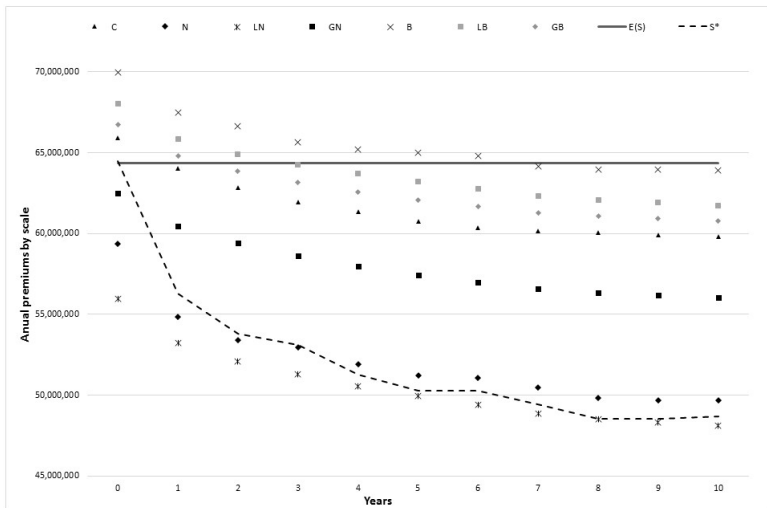
j	C	N	NL	NG	B	LB	GB	N. Policies	$\hat{\lambda}_j$
1	45	33.4	32.4	41.0	48.8	46.0	45.7	174,173	0.034516
2	45	48.0	39.9	45.3	58.0	52.0	49.8	109,113	0.072883
3	50	49.5	47.4	50.0	60.1	58.0	54.4	42,736	0.076425
4	55	51.1	54.9	55.2	62.3	64.0	59.3	29,134	0.080265
5	60	66.5	62.4	61.0	63.7	70.0	64.7	23,730	0.126855
6	65	69.9	69.9	67.3	66.2	76.0	70.6	4,241	0.135954
7	70	74.6	77.5	74.3	68.1	82.0	77.0	2,759	0.148393
8	80	87.3	85.0	82.0	69.9	88.0	84.0	24,829	0.181802
9	90	92.9	92.5	90.6	72.3	94.0	91.7	11,747	0.195919
10	100	100.0	100.0	100.0	100.0	100.0	100.0	166	0.213730
11	110	109.8	107.5	110.4	105.6	106.0	109.1	2,882	0.237433
12	120	117.0	115.0	121.9	113.0	112.0	119.0	7,632	0.255984
13	130	125.3	122.5	134.6	124.3	118.0	129.9	250	0.277505
14	150	134.5	130.1	148.6	148.8	124.0	141.7	710	0.301956
15	180	143.4	137.6	164.0	162.6	130.0	154.6	2,256	0.327931
16	250	153.3	145.1	181.1	181.9	136.0	168.6	2,643	0.358676
17	325	164.1	152.6	199.9	209.1	142.0	184.0	1,304	0.395719
18	400	176.0	160.1	220.7	235.0	148.0	200.7	2,183	0.441571
								442,490	

Table: N. policies, Poisson parameter and bonus scales by class

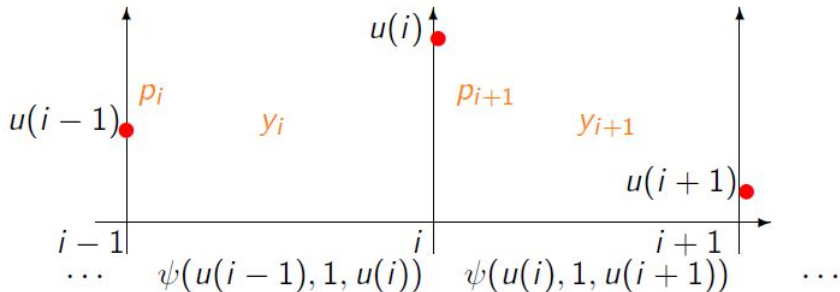
$j \setminus \text{Year}$	0	2	5	10	stationarity
1	0.39362	0.63576	0.67436	0.71953	0.73121
2	0.24659	0.05681	0.04089	0.05678	0.04913
3	0.09658	0.06896	0.08045	0.04929	0.05394
4	0.06584	0.05938	0.06980	0.04927	0.05941
5	0.05363	0.01651	0.01678	0.01904	0.01871
6	0.00958	0.05774	0.01747	0.02528	0.01678
7	0.00623	0.03644	0.03326	0.01645	0.01411
8	0.05611	0.00444	0.01265	0.01037	0.00837
9	0.02655	0.00793	0.00659	0.00840	0.00725
10	0.00037	0.02375	0.00878	0.00977	0.00612
11	0.00651	0.00480	0.01277	0.00576	0.00495
12	0.01725	0.00188	0.00596	0.00451	0.00453
13	0.00056	0.00620	0.00541	0.00407	0.00419
14	0.00160	0.00818	0.00344	0.00435	0.00398
15	0.00510	0.00294	0.00403	0.00332	0.00398
16	0.00597	0.00468	0.00208	0.00326	0.00410
17	0.00295	0.00195	0.00210	0.00429	0.00438
18	0.00493	0.00164	0.00317	0.00629	0.00488

Table: Portfolio distribution over time and classes

Expected Premium's behavior (P=115,838,792)



One run



Probability of ruin given a initial surplus u

u	t	P_0	$\hat{\psi}(u, t)$ (%)						
			C	N	NL	NG	B	BL	BG
350,000	2	1.246	60.502	99.999	100	96.129	25.665	38.028	50.392
	5	1.246	60.502	99.999	100	96.129	25.665	38.028	50.392
	10	1.246	60.502	99.999	100	96.129	25.665	38.028	50.392
1,500,000	2	0	14.807	99.954	100	75.749	0.953	3.407	8.374
	5	0	14.807	99.969	100	75.749	0.953	3.407	8.374
	10	0	14.807	99.969	100	75.749	0.953	3.407	8.374
2,000,000	2	0	7.569	99.854	100	63.277	0.233	1.186	3.716
	5	0	7.569	99.902	100	63.277	0.233	1.186	3.716
	10	0	7.569	99.902	100	63.277	0.233	1.186	3.716
2,550,000	2	0	3.495	99.601	100	48.724	0.052	0.369	1.463
	5	0	3.495	99.716	100	48.724	0.052	0.369	1.463
	10	0	3.495	99.717	100	48.724	0.052	0.369	1.463
3,250,000	2	0	1.206	98.790	100	31.551	0.009	0.083	0.426
	5	0	1.206	99.150	100	31.551	0.009	0.083	0.426
	10	0	1.206	99.154	100	31.551	0.009	0.083	0.426
6,400,000	2	0	0.008	74.639	99.923	1.245	0	0	0.003
	5	0	0.008	81.050	99.986	1.245	0	0	0.003
	10	0	0.008	81.163	99.989	1.245	0	0	0.003
15,000,000	2	0	0	0.360	31.246	0	0	0	0
	5	0	0	1.580	70.362	0	0	0	0
	10	0	0	1.623	74.868	0	0	0	0
25,000,000	2	0	0	0	0.002	0	0	0	0
	5	0	0	0	0.879	0	0	0	0
	10	0	0	0	2.037	0	0	0	0

Table: $n = 2, 5$ and 10 years for each bonus scale (results in percentages)

Notes:

- Recall: The portfolio is closed for ingoing and outgoing of policyholders;
- The introduction of a BMS **increases the ruin probabilities** but, with our model, we can estimate the magnitude of that increasing;
- No perceptible changes in the scales can have a **big impact on initial surplus** u in order to have an acceptable ruin probability.

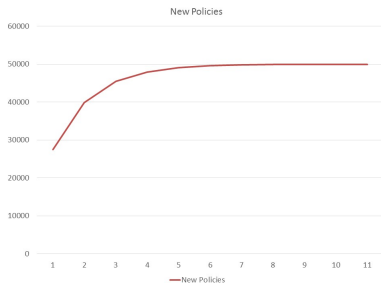
Open Model for SBM

k	ingoing	outgoing
1	0,26585	0,046442
2	0,08396	0,056989
3	0,03745	0,056703
4	0,08933	0,074157
5	0,06386	0,070393
6	0,16659	0,08804
7	0,10947	0,100813
8	0,09598	0,109777
9	0,03959	0,147588
10	0,045	0,20866
11	0,00176	0,380737
12	0,00097	0,388989
13	0,00018	0,397241
14	0,00003	0,487619
15	0	0,497778
16	0	0,098462
17	0	0,087521
18	0	0,068072

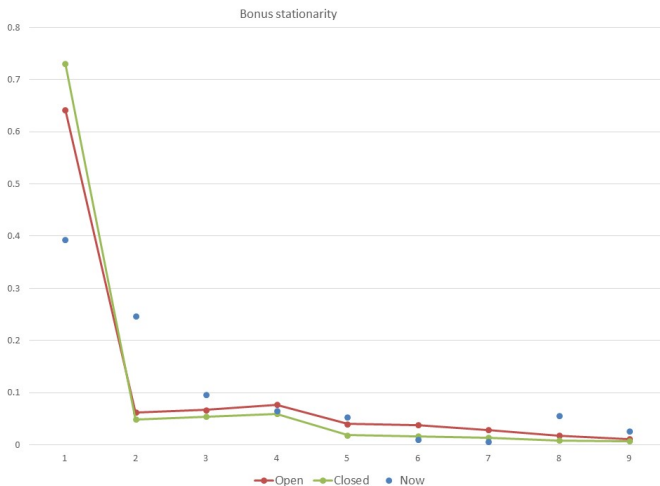
Number of new policies in year i

$$\tau(1 - e^{-\delta i})$$

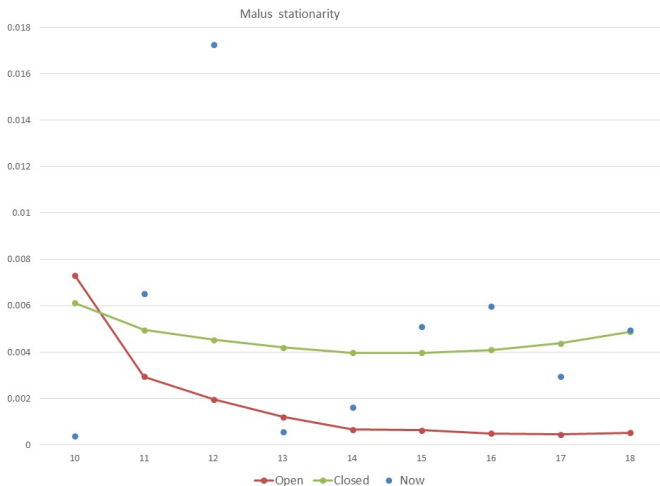
$$\tau = 50000, \delta = 0.8$$



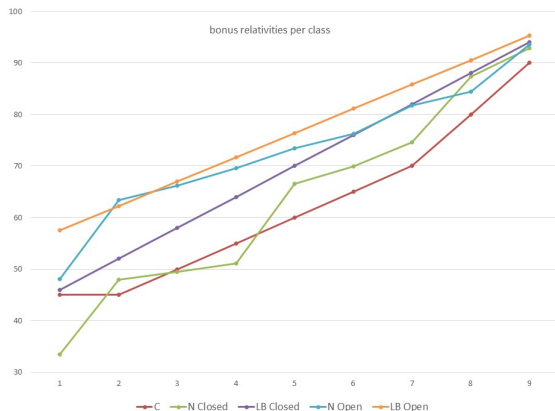
Portfolio distribution (stationarity) over classes - bonus



Portfolio distribution (stationarity) over classes - malus

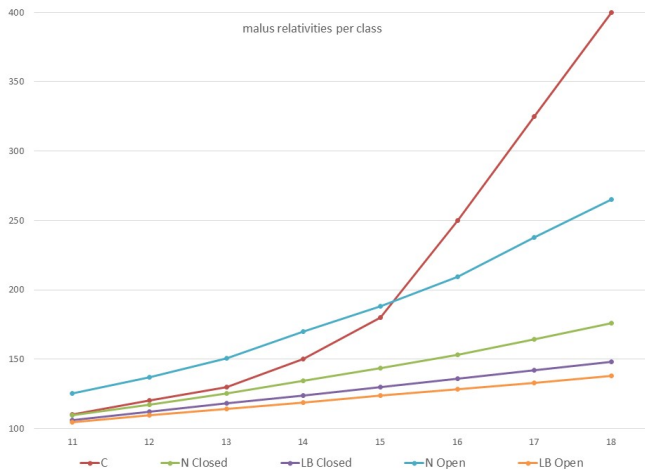


Relativities - bonus ¹

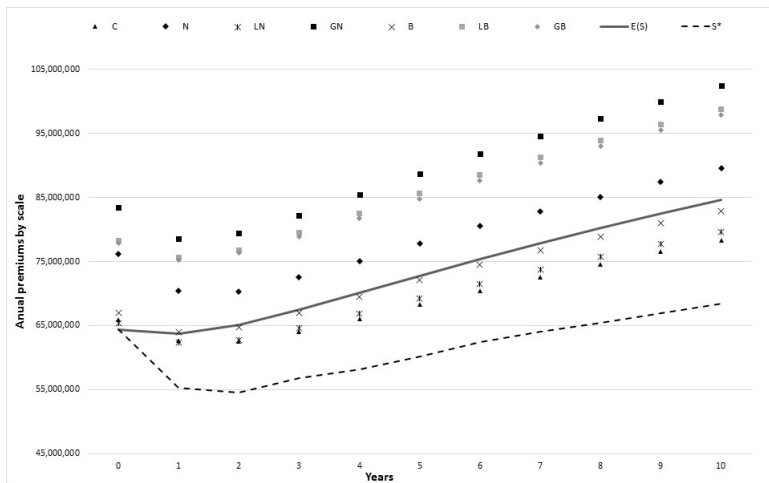


¹Relativities calculated considering an Open Model for SBM and kept constant during 10 years.

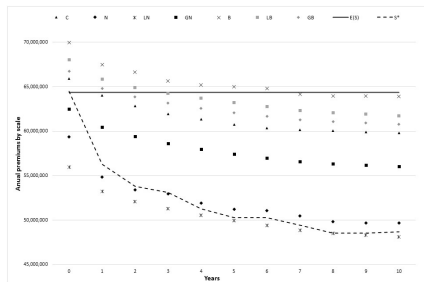
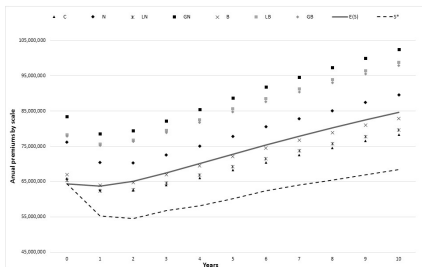
Relativities - malus



Expected Premium's behavior



Expected Premium's behavior open vs classical model for SBM



Probability of ruin given a initial surplus u - Open Model for
SBM

$\widehat{\psi}(u, t = 10)$ (%)						
		Open Model			Classical Model	
u	P_0	C	N	LB	N	LB
350.000	1,243	60,436	10,265	8,172	100	37,894
1.500.000	0	14,752	0,055	0,029	99,991	3,298
2.000.000	0	7,482	0,006	0,002	99,967	1,116
2.550.000	0	3,334	0	0	99,896	0,328
3.250.000	0	1,097	0	0	99,653	0,065
6.400.000	0	0,003	0	0	91,621	0
15.000.000	0	0	0	0	17,378	0
25.000.000	0	0	0	0	0,149	0

Notes:

- The model may also be useful as a way to **estimate the amount needed for initial capital u** .
- The model provides simple and effective methodology for **assessing scales and bonus malus schemes**.
- It can be applied also for the Solvency II purposes to obtain the estimates of ruin probabilities in **one year period**.

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