Convex ordering for insurance preferences

Alfred Chong

Department of Statistics and Actuarial Science, The University of Hong Kong

Department of Mathematics, King's College London

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Joint work with Ka Chun Cheung (The University of Hong Kong) and Phillip Yam (The Chinese University of Hong Kong)

Problem Formulation

Let X be a non-negative random variable to model an insurable loss of an agent. After purchasing an insurance I, the policyholder retains a part of the loss $R(X) \triangleq X - I(X)$, and pays a premium $P(I(X))$. The total loss exposure of the policyholder is

$$L_I(X) \triangleq X - I(X) + P(I(X)) = R(X) + P(I(X)).$$

Problem Formulation (cont.)

Consider the admissible set $\mathcal{I}$ of ceded loss functions I:

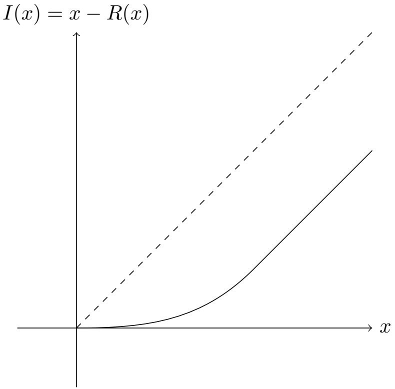
$$\begin{aligned}\mathcal{I} \triangleq \{I : [0, \operatorname{ess\,sup} X] \to [0, \operatorname{ess\,sup} X] |\\ &(I1) : 0 \le I(x) \le x, \quad \text{for any } x \in [0, \operatorname{ess\,sup} X];\\ &(I2) : 0 \le I(x_1) - I(x_2) \le x_1 - x_2, \quad \text{for any } 0 \le x_2 < x_1 \le \operatorname{ess\,sup} X\}.\end{aligned}$$

Also, consider the admissible set $\mathcal{R}$ of retained loss functions $R \triangleq Id - I$:

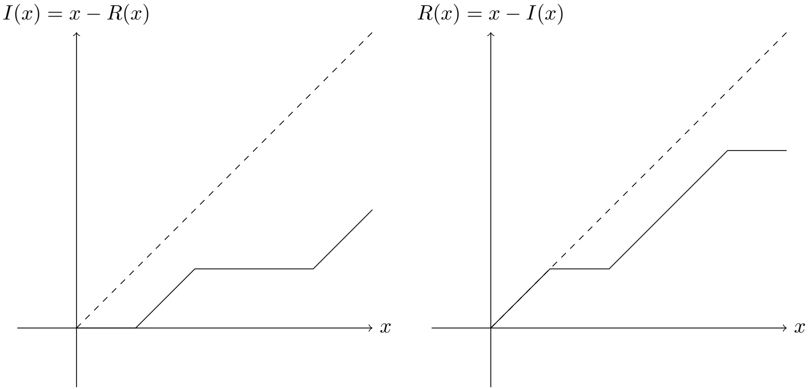
$$\begin{aligned}\mathcal{R} \triangleq \{R : [0, \operatorname{ess\,sup} X] \to [0, \operatorname{ess\,sup} X] |\\ &(R1) : 0 \le R(x) \le x, \quad \text{for any } x \in [0, \operatorname{ess\,sup} X];\\ &(R2) : 0 \le R(x_1) - R(x_2) \le x_1 - x_2, \quad \text{for any } 0 \le x_2 < x_1 \le \operatorname{ess\,sup} X\}.\end{aligned}$$

Both admissible I and R are non-decreasing.

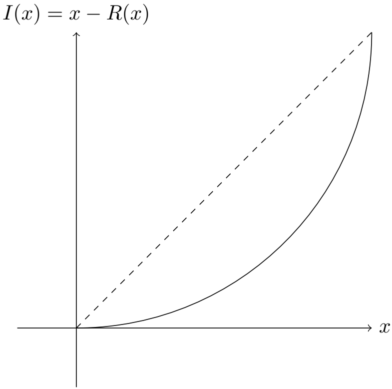
Admissible I and R

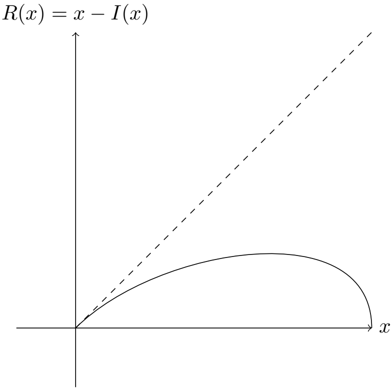


Admissible I and R



Non-admissible I and R





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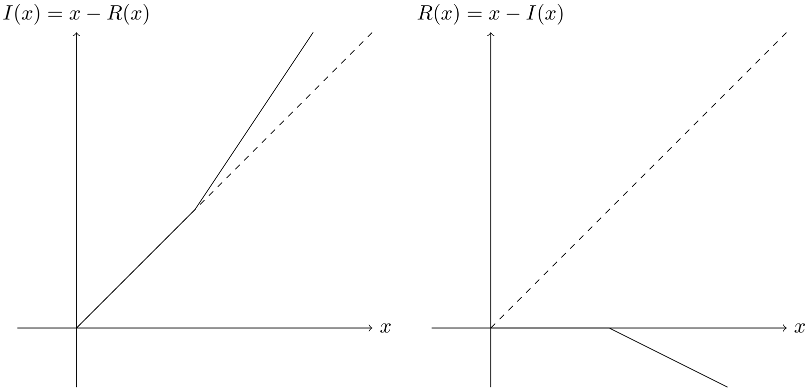


Problem Formulation (cont.)

Problem: Let G be a non-decreasing (in all arguments) function from $\mathbb{R}_+^3$ to $\mathbb{R}$. Let g be a non-decreasing (in both arguments) function from $\mathbb{R}_+^2$ to $\mathbb{R}_+$. Let V_{cx} be a functional preserving convex order. Let $\alpha \in (0, 1]$. The policyholder aims to choose an optimal insurance indemnity $I \in \mathcal{I}$ that minimizes

$$G(\mathbb{E}\left[L_I(X)\right], \mathsf{V@R}_\alpha(L_I(X)), \mathsf{TV@R}_\alpha(L_I(X))),$$

with

$$P(I(X)) = g\left(\mathbb{E}[I(X)], V_{\mathsf{cx}}(I(X))\right).$$

The premium principle includes the expected value principle, Wang's principle, Dutch principle, variance principle, and standard deviation principle, etc.

Preliminaries

- The (left-continuous) quantile function of Y is defined by

$$F_Y^{-1}(p) \triangleq \inf\{y \in \mathbb{R} | F_Y(y) \geq p\}, \quad \forall p \in (0,1].$$

- The Value-at-Risk of Y at the level $\alpha \in (0,1]$, denoted by $\text{V@R}_\alpha(Y)$, is defined by

$$\text{V@R}_\alpha(Y) \triangleq F_Y^{-1}(\alpha).$$

- The Tail Value-at-Risk of Y at the level $\alpha \in (0,1]$, denoted by $\text{TV@R}_\alpha(Y)$, is defined by

$$\text{TV@R}_\alpha(Y) \triangleq \frac{1}{1-\alpha}\int_\alpha^1 \text{V@R}_\lambda(Y)d\lambda = \frac{1}{1-\alpha}\int_\alpha^1 F_Y^{-1}(\lambda)d\lambda,$$

for $\alpha < 1$; when $\alpha = 1$, $\text{TV@R}_\alpha(Y)$ is defined by the $\operatorname{ess\,sup} Y$.

- The stop-loss transform of Y is defined by

$$\pi_Y(t) \triangleq \mathbb{E}[(Y-t)_+] = \int_t^{\operatorname{ess\,sup} Y} S_Y(y)dy, \quad \forall t \in \mathbb{R}.$$

- Y and Z are comonotonic if there exist a random variable S and two non-decreasing functions f_1 and f_2 such that $Y \overset{d}{=} f_1(S)$ and $Z \overset{d}{=} f_2(S)$.

Preliminaries (cont.)

Proposition 1

Assume that $\alpha \in (0,1]$.

(i) Both V@R and TV@R are translation invariant and comonotonic additive, i.e., for comonotonic Y and Z, and $c \in \mathbb{R}$,

$$\text{(T)V@R}_\alpha(Y+c) = \text{(T)V@R}_\alpha(Y) + c;$$

$$\text{(T)V@R}_\alpha(Y+Z) = \text{(T)V@R}_\alpha(Y) + \text{(T)V@R}_\alpha(Z).$$

(ii) For any non-decreasing continuous function f,

$$\text{V@R}_\alpha(f(Y)) = f(\text{V@R}_\alpha(Y)).$$

(iii) For $\alpha \in (0,1)$,

$$\text{TV@R}_\alpha(Y) = \text{V@R}_\alpha(Y) + \frac{1}{1-\alpha}\pi_Y(\text{V@R}_\alpha(Y));$$

for $\alpha = 1$,

$$\text{TV@R}_\alpha(Y) = \text{V@R}_\alpha(Y) = \operatorname{ess\,sup} X.$$

Preliminaries (cont.)

By (i) in Proposition 1,

$$\mathbb{E}[L_I(X)] = \mathbb{E}[X - I(X) + P(I(X))] = \mathbb{E}[X] - \mathbb{E}[I(X)] + P(I(X)).$$

$$\mathsf{V@R}_\alpha(L_I(X)) = \mathsf{V@R}_\alpha(R(X) + P(I(X))) = \mathsf{V@R}_\alpha(X) - \mathsf{V@R}_\alpha(I(X)) + P(I(X)).$$

$$\mathsf{TV@R}_\alpha(L_I(X)) = \mathsf{TV@R}_\alpha(R(X)+P(I(X))) = \mathsf{TV@R}_\alpha(X) - \mathsf{TV@R}_\alpha(I(X)) + P(I(X)).$$

Hence, the problem is equivalent to

$$\begin{aligned}\inf_{I\in\mathcal{I}} G\{&\mathbb{E}[X] - \mathbb{E}[I(X)] + g\left(\mathbb{E}[I(X)], V_{\mathsf{cx}}(I(X))\right),\\ &\mathsf{V@R}_\alpha(X) - \mathsf{V@R}_\alpha(I(X)) + g\left(\mathbb{E}[I(X)], V_{\mathsf{cx}}(I(X))\right),\\ &\mathsf{TV@R}_\alpha(X) - \mathsf{TV@R}_\alpha(I(X)) + g\left(\mathbb{E}[I(X)], V_{\mathsf{cx}}(I(X))\right)\}.\end{aligned}$$

Preliminaries (cont.)

- Y is smaller than Z in convex order sense if, for all convex functions $\varphi : \mathbb{R} \to \mathbb{R}$, $\mathbb{E}[\varphi(Y)] \leq \mathbb{E}[\varphi(Z)]$. The convex ordering is denoted by $Y \leq_{\text{cx}} Z$.
- V preserves convex order if $V(Y) \leq V(Z)$ for $Y \leq_{\text{cx}} Z$.
- The distribution functions F_Y and F_Z cross $k \geq 1$ times if there exist

$$\xi_{0,2} < \xi_{1,1} \leq \xi_{1,2} < \xi_{2,1} \leq \xi_{2,2} < \cdots < \xi_{k,1} \leq \xi_{k,2} < \xi_{k+1,1},$$

where

$$\xi_{0,2} \triangleq \inf\{x : F_Y(x) \neq F_Z(x)\},$$

and

$$\xi_{k+1,1} \triangleq \sup\{x : F_Y(x) \neq F_Z(x)\},$$

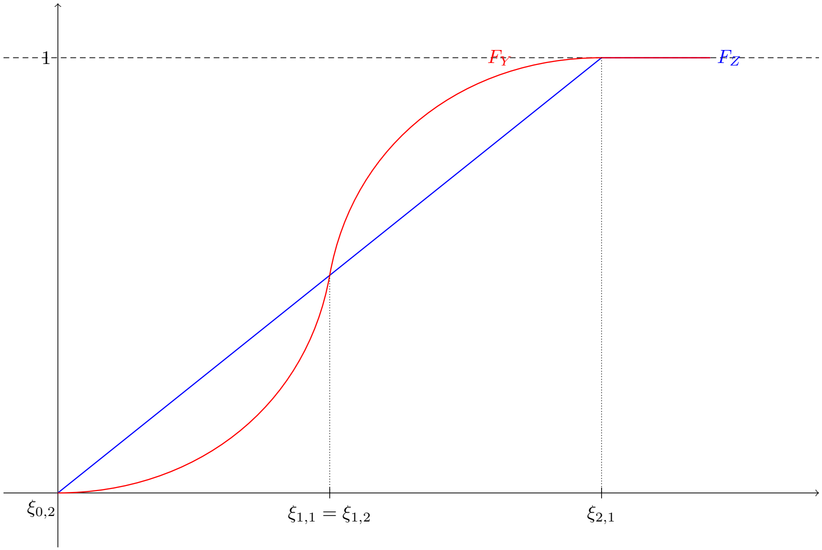
such that, for each $i = 1, 2, \ldots, k$,

(i) for any $y \in (\xi_{i-1,2}, \xi_{i,1})$ and $z \in (\xi_{i,2}, \xi_{i+1,1})$,

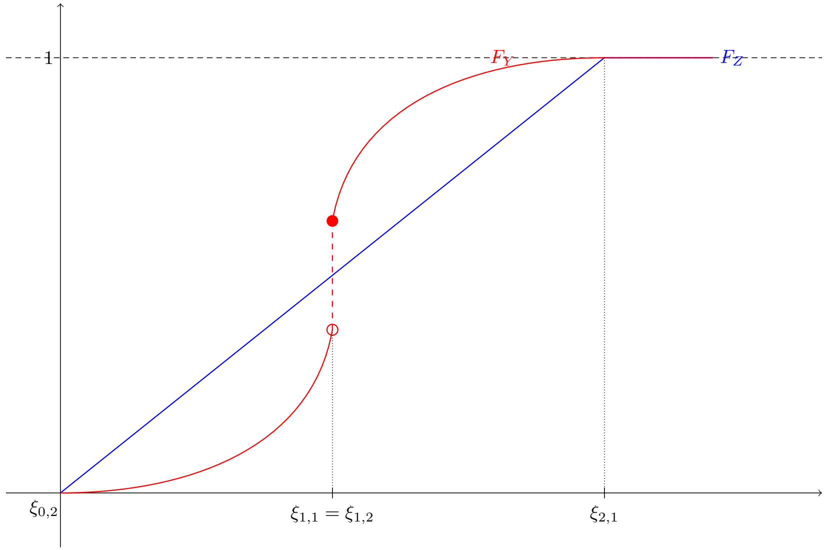
$$(F_Y(y) - F_Z(y))\,(F_Y(z) - F_Z(z)) < 0; \text{ and}$$

(ii) if $\xi_{i,1} < \xi_{i,2}$, $F_Y(z) = F_Z(z)$ for any $\xi_{i,1} \leq z < \xi_{i,2}$.

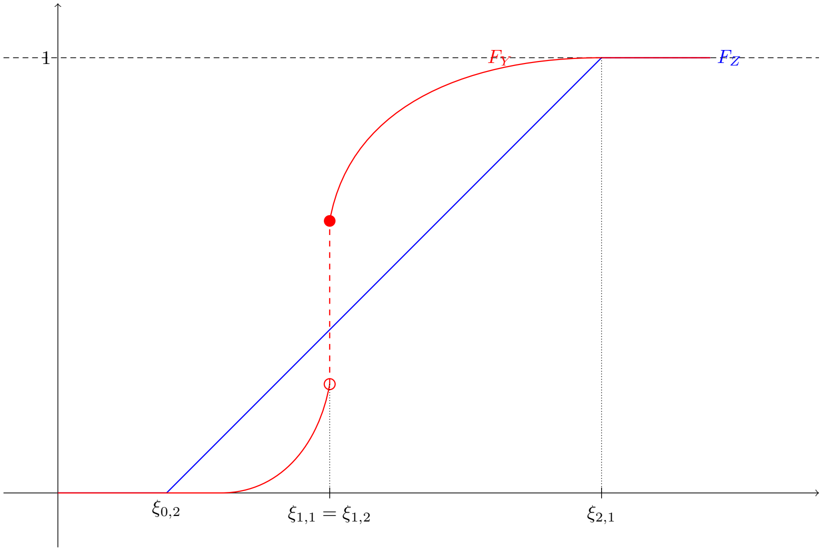
Crossing Once



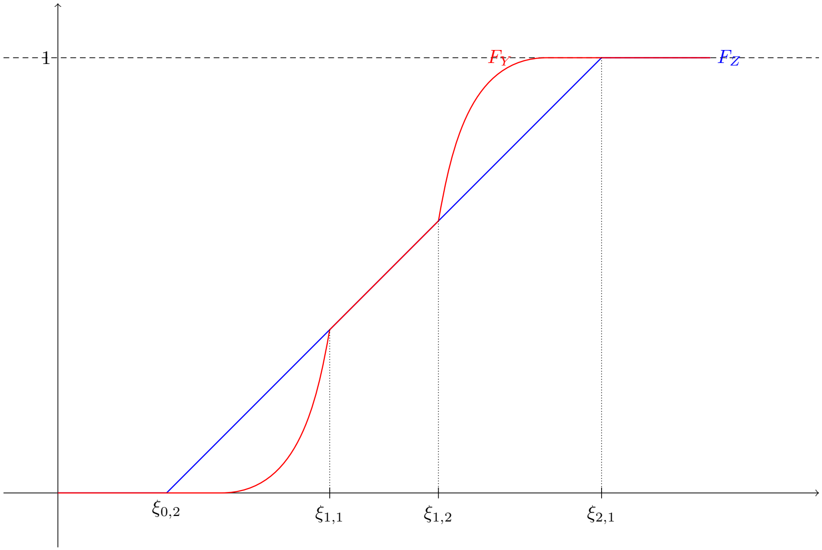
Crossing Once with Jump



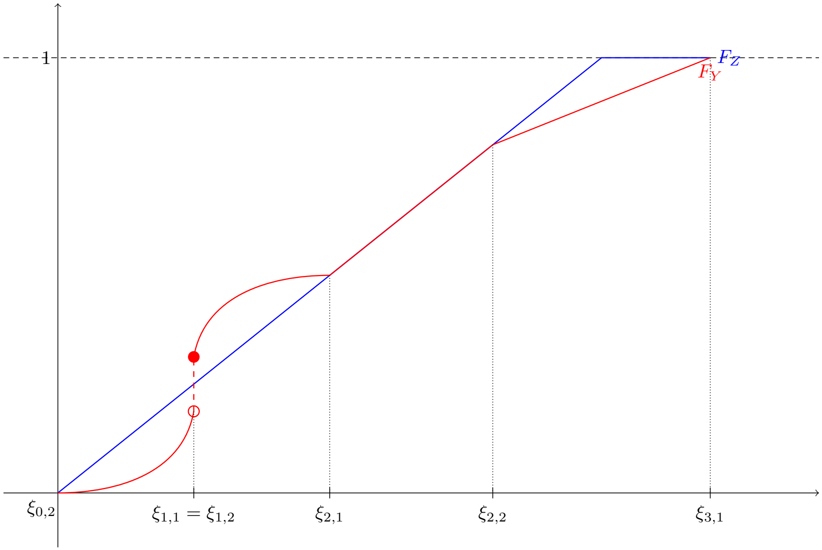
Crossing Once with Jump



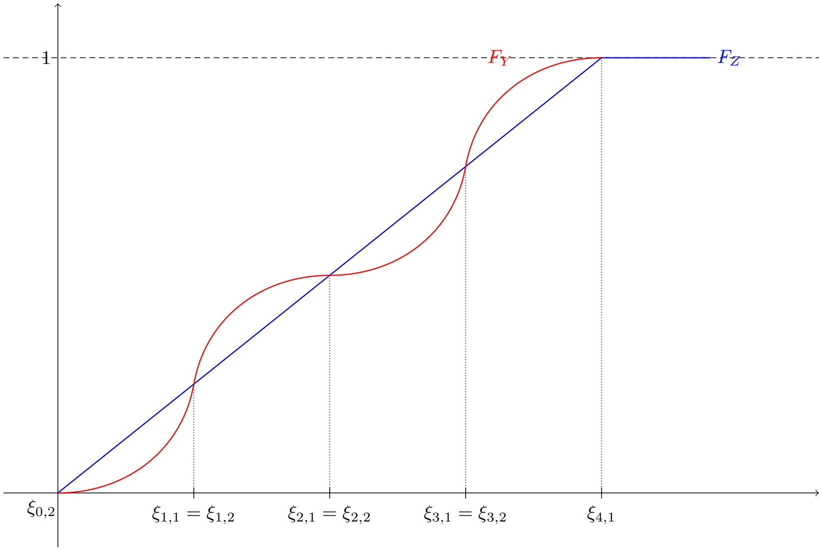
Crossing Once with Overlapping



Crossing Twice with Jump and Overlapping



Crossing Thrice



Preliminaries (cont.)

Theorem 2 (Karlin-Novikoff once-crossing conditions)

Assume that F_Y and F_Z cross once. Then $Y \leq_{cx} Z$ if, and only if,

(i) $\mathbb{E}[Y] = \mathbb{E}[Z]$; *and*

(ii) $F_Y(x) < F_Z(x)$ *for* $\xi_{0,2} < x < \xi_{1,1}$.

Theorem 3 (Karlin-Novikoff-Stoyan-Taylor crossing conditions)

Assume that F_Y and F_Z cross $n \geq 1$ times. Then $Y \leq_{cx} Z$ if, and only if,

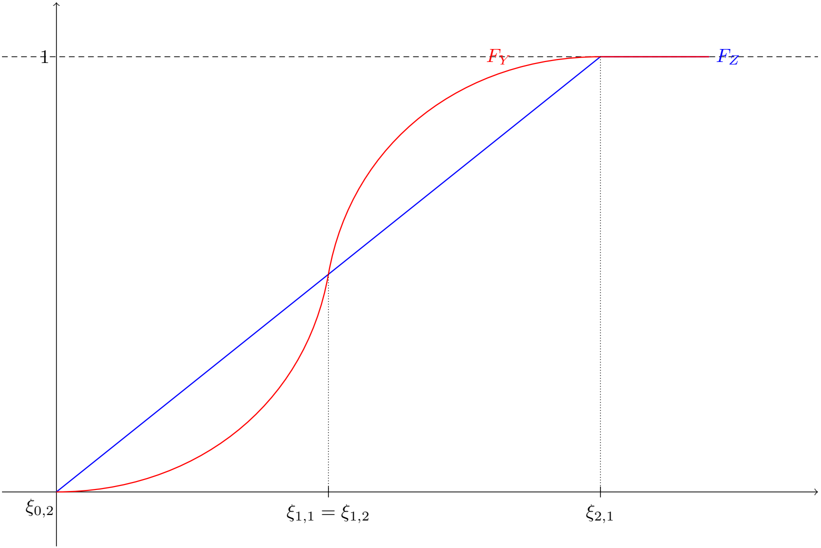
(i) $\mathbb{E}[Y] = \mathbb{E}[Z]$;

(ii) $F_Y(x) < F_Z(x)$ *for* $\xi_{0,2} < x < \xi_{1,1}$;

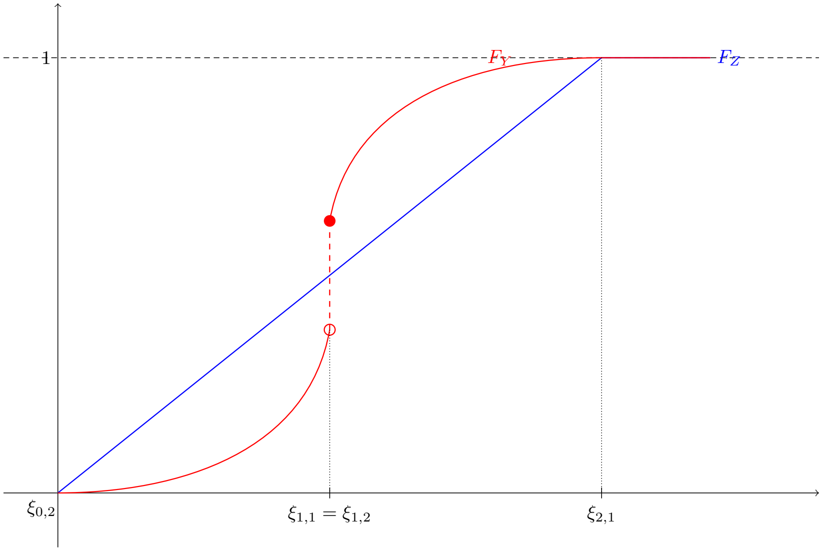
(iii) *there is an odd number of crossings* $n = 2m - 1$ *where* $m = 1, 2, \ldots$; *and*

(iv) *if* $m \geq 2$, *for any* $j = 1, 2, \ldots, m - 1$, $\pi_Y(\xi_{2j,2}) \leq \pi_Z(\xi_{2j,2})$.

$Y \leq_{\text{cx}} Z$



$Y \leq_{\mathsf{cx}} Z$



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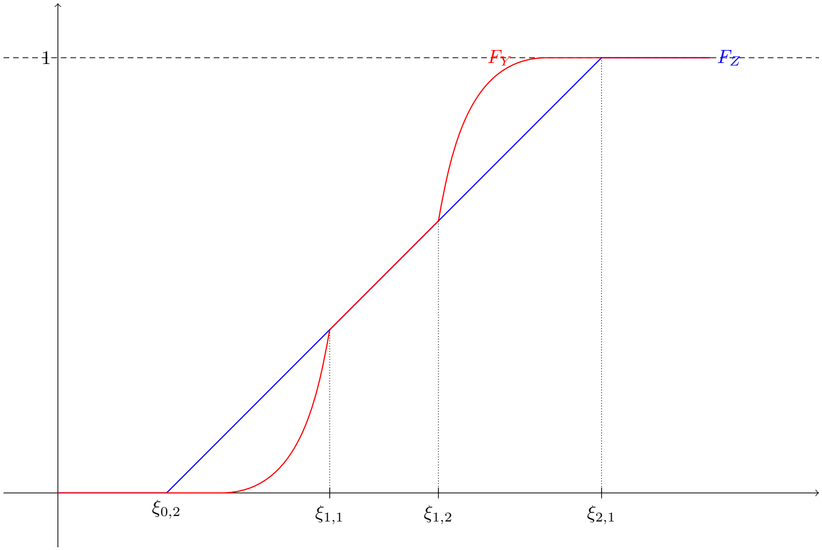

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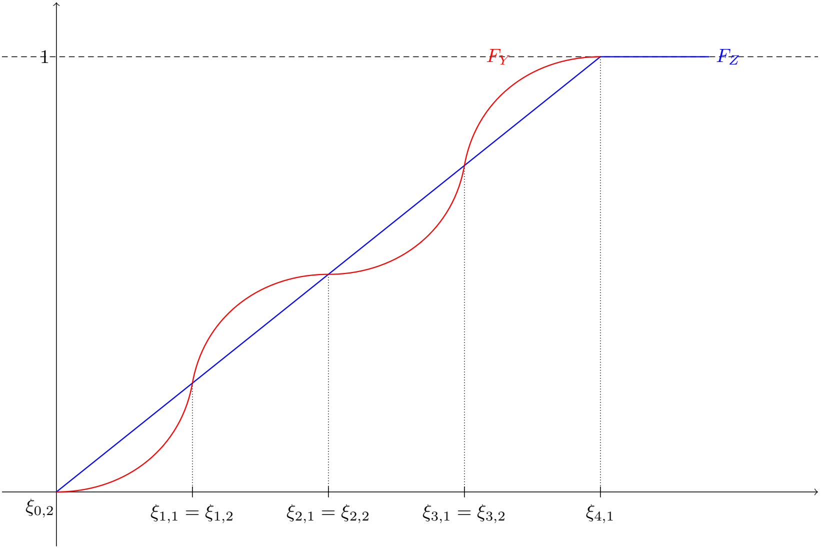
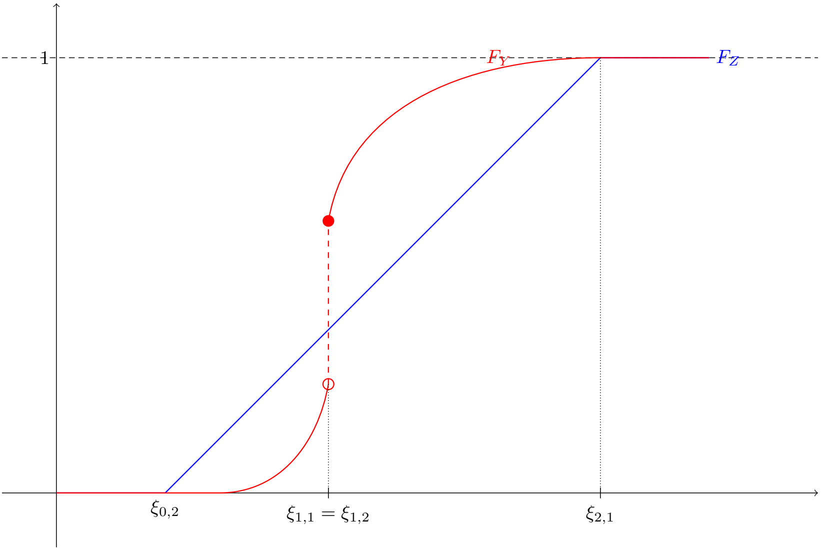


Not $Y \leq_{\text{cx}} Z$



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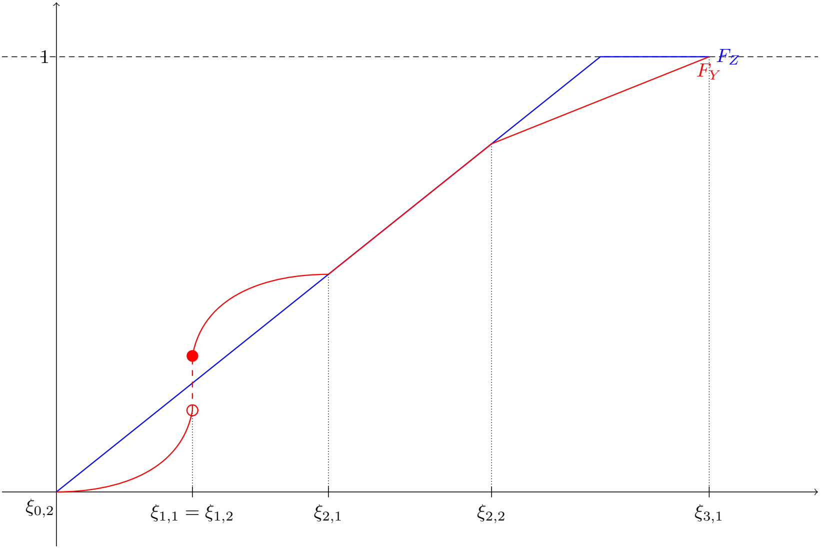


Distribution Function of $I(X)$ for Admissible I

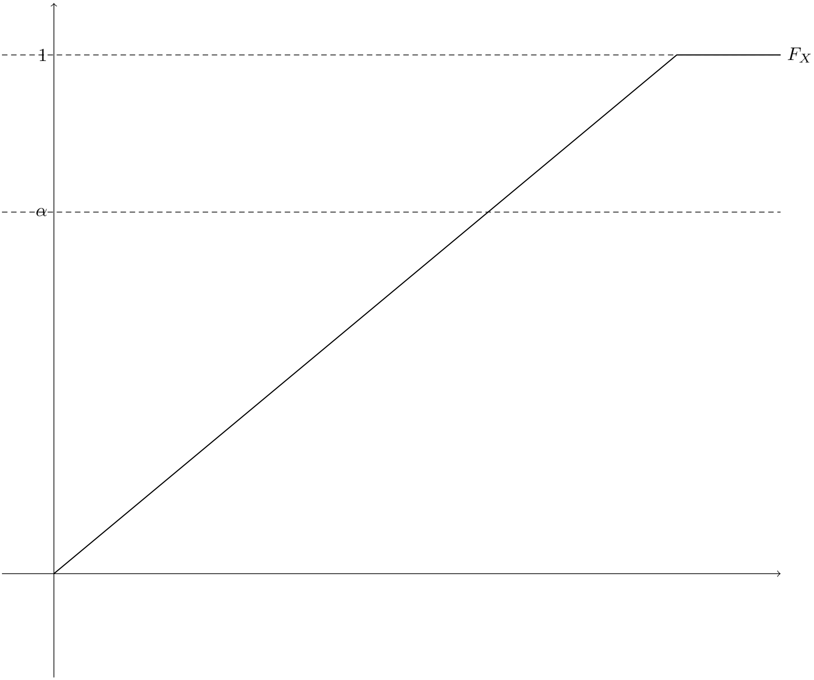
Conditions $(I1)$ and $(I2)$ in $\mathcal{I}$ can be expressed in terms of the distribution function of $I(X)$:

(i) $F_{I(X)}(0-) = 0$ and $F_X(z) \leq F_{I(X)}(z)$ for any $z \in [0, \operatorname{ess\,sup} X]$;

(ii) $0 \leq F^{-1}_{I(X)}(\beta_2) - F^{-1}_{I(X)}(\beta_1) \leq F^{-1}_X(\beta_2) - F^{-1}_X(\beta_1)$ for any $0 \leq \beta_1 < \beta_2 \leq 1$.

Recall that the problem is

$$\inf_{I \in \mathcal{I}} G\{\mathbb{E}\left[X\right] - \mathbb{E}\left[I(X)\right] + g\left(\mathbb{E}[I(X)], V_{\mathsf{cx}}(I(X))\right),$$
$$\mathsf{V@R}_\alpha(X) - \mathsf{V@R}_\alpha(I(X)) + g\left(\mathbb{E}[I(X)], V_{\mathsf{cx}}(I(X))\right),$$
$$\mathsf{TV@R}_\alpha(X) - \mathsf{TV@R}_\alpha(I(X)) + g\left(\mathbb{E}[I(X)], V_{\mathsf{cx}}(I(X))\right)\}.$$



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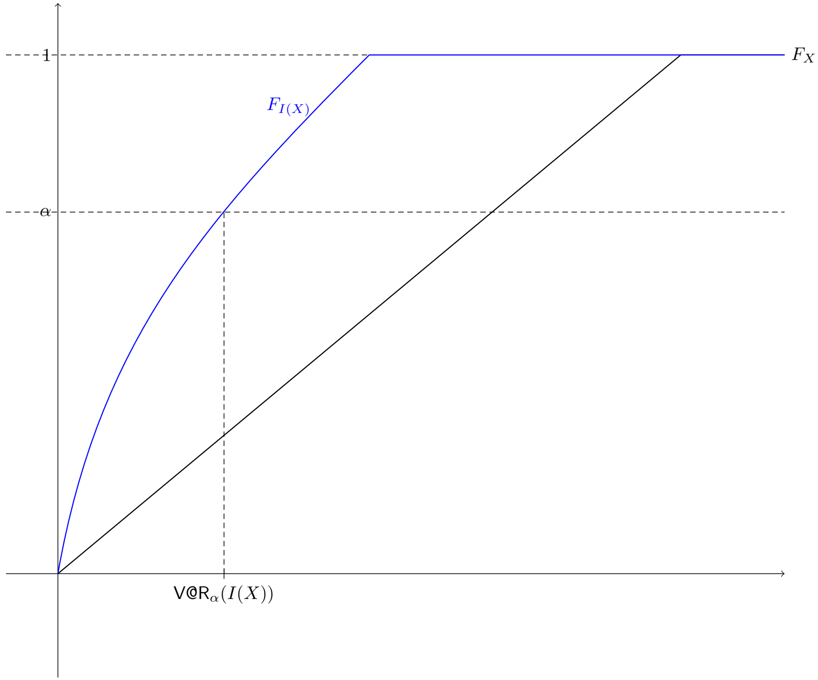

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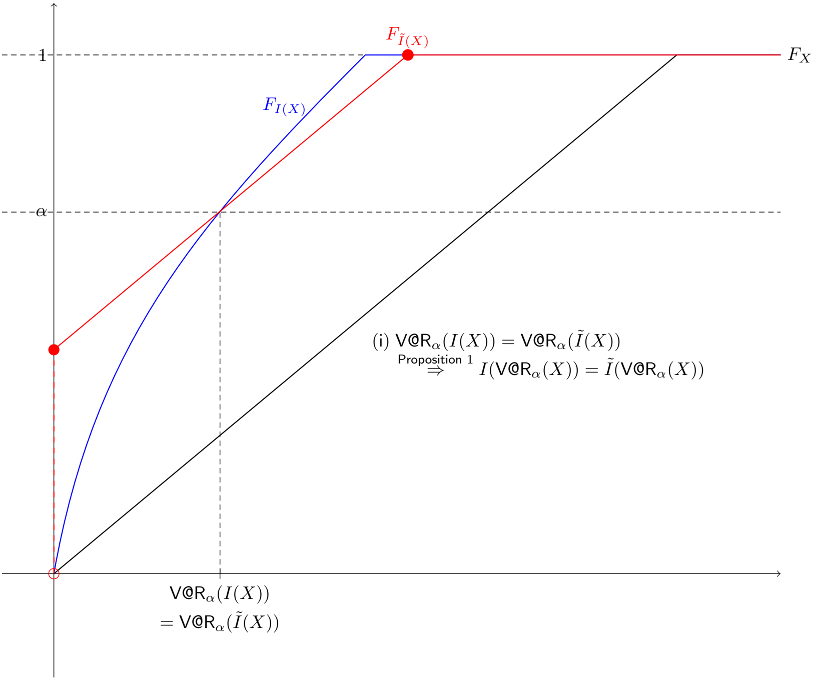
$F_{I(X)}$
$\mathsf{V@R}_\alpha(I(X))$



$F_{\tilde{I}(X)}$
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$F_{I(X)}$
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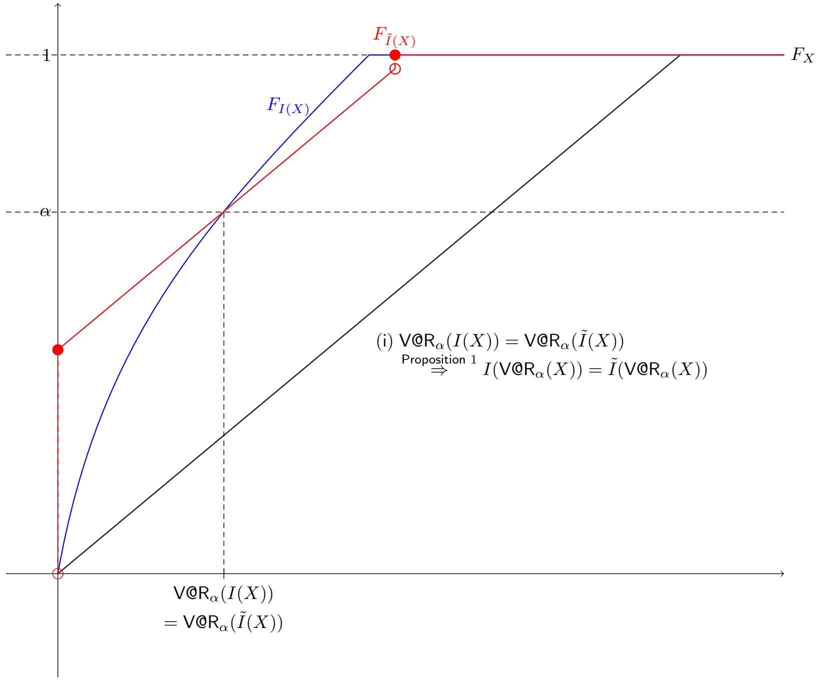
(i) $\mathsf{V@R}_\alpha(I(X)) = \mathsf{V@R}_\alpha(\tilde{I}(X))$
$\overset{\text{Proposition 1}}{\Rightarrow} I(\mathsf{V@R}_\alpha(X)) = \tilde{I}(\mathsf{V@R}_\alpha(X))$
$\mathsf{V@R}_\alpha(I(X))$
$= \mathsf{V@R}_\alpha(\tilde{I}(X))$



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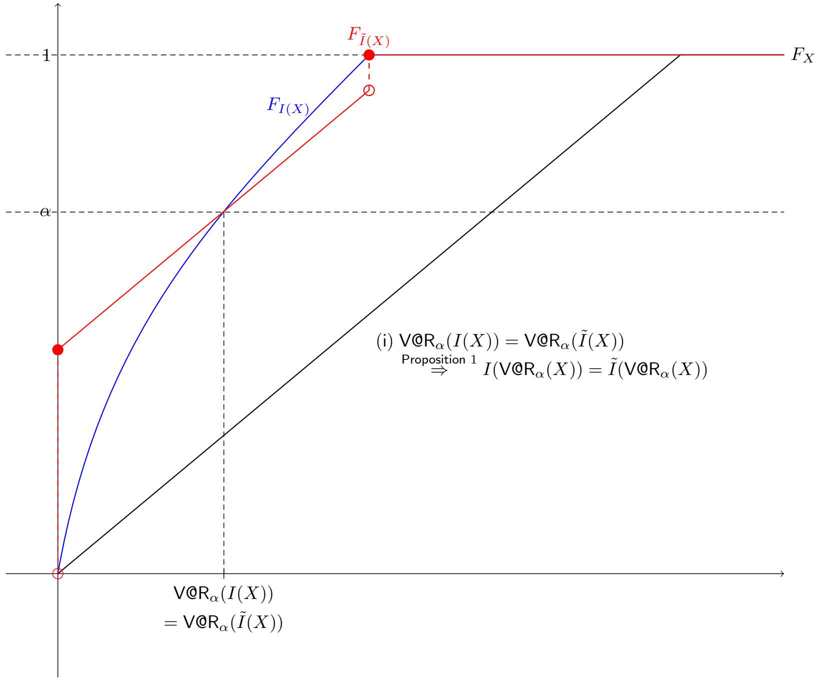

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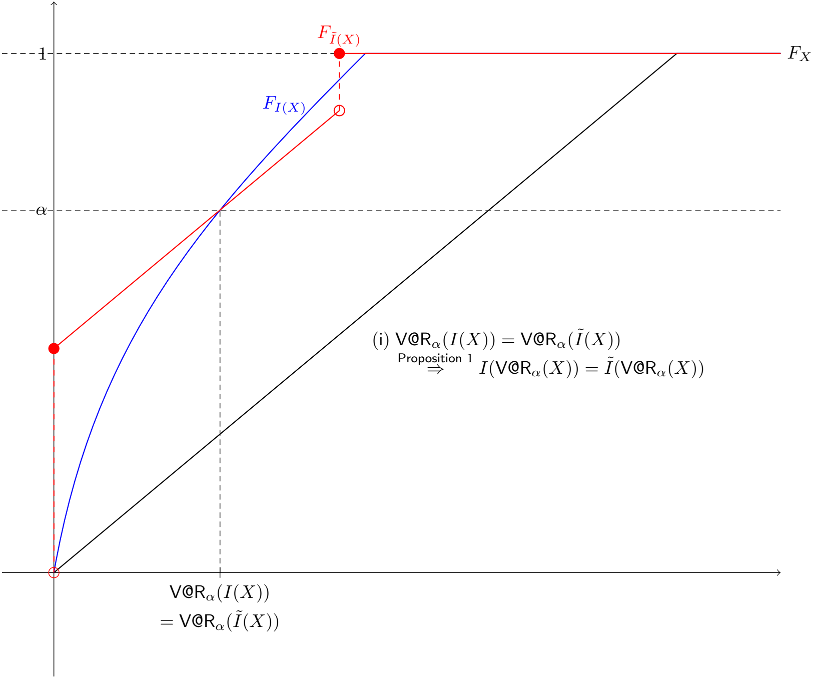

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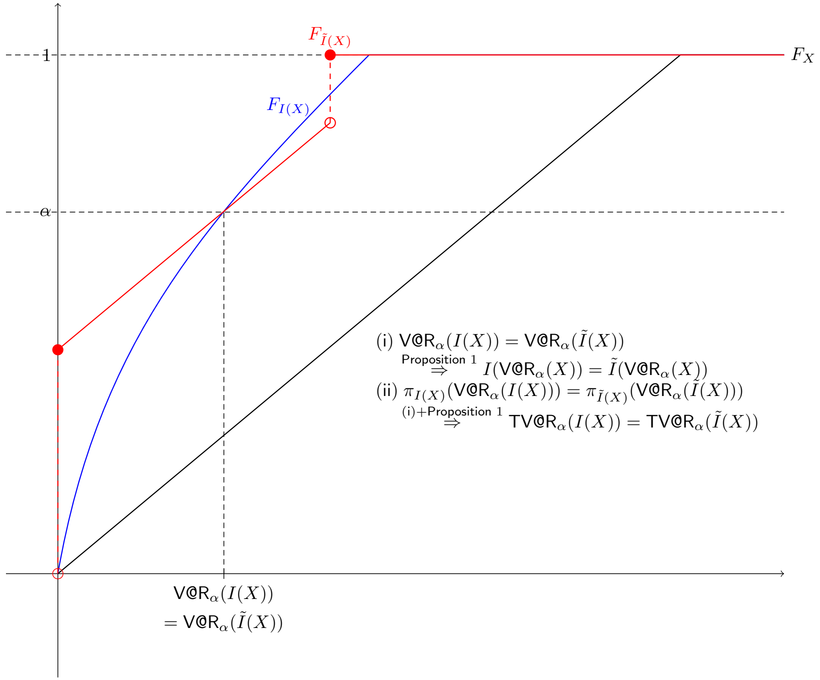
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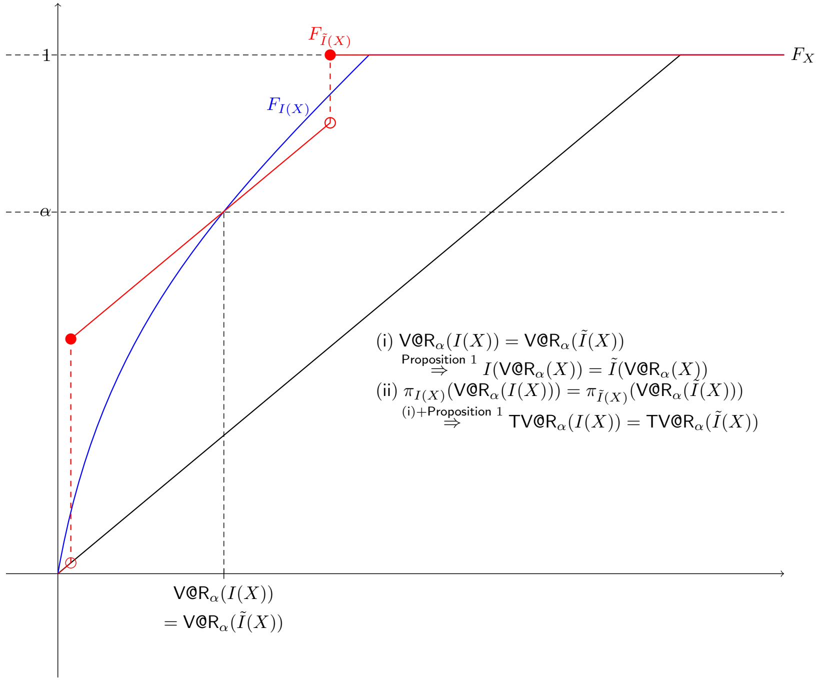
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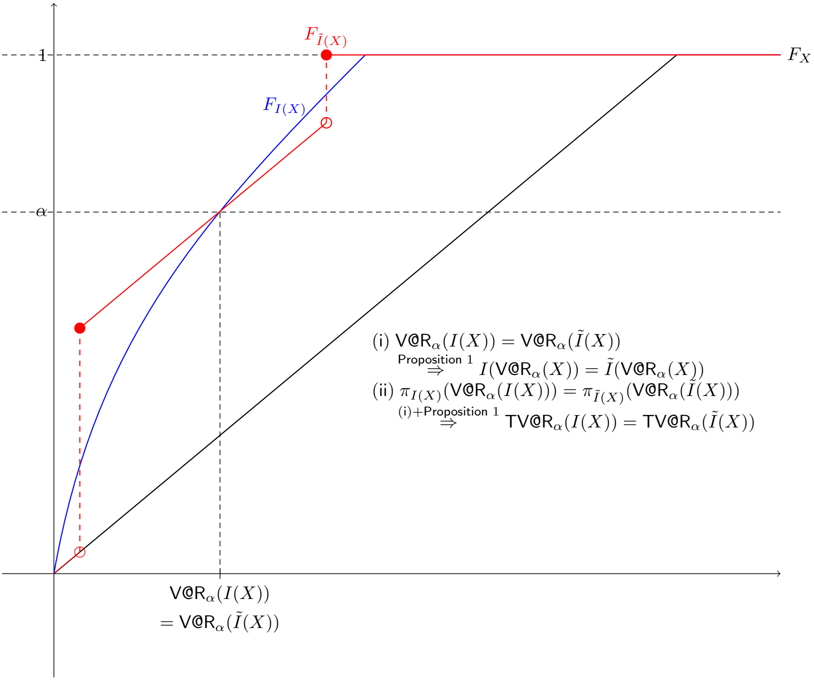

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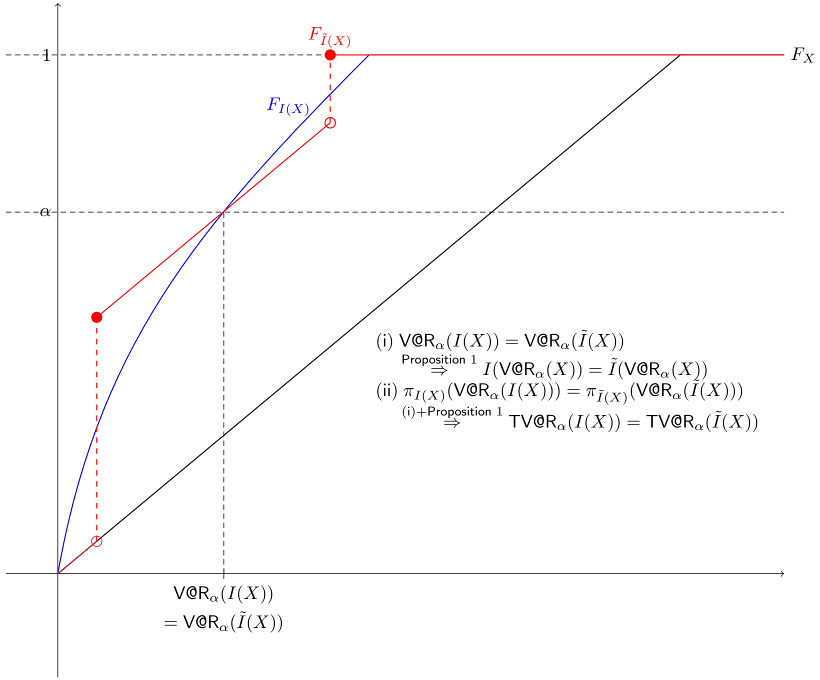

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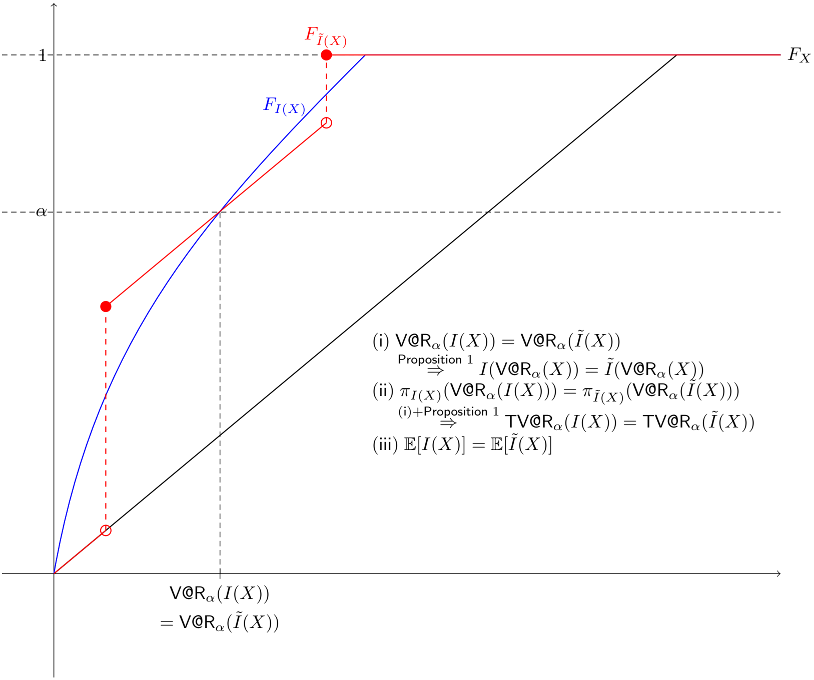

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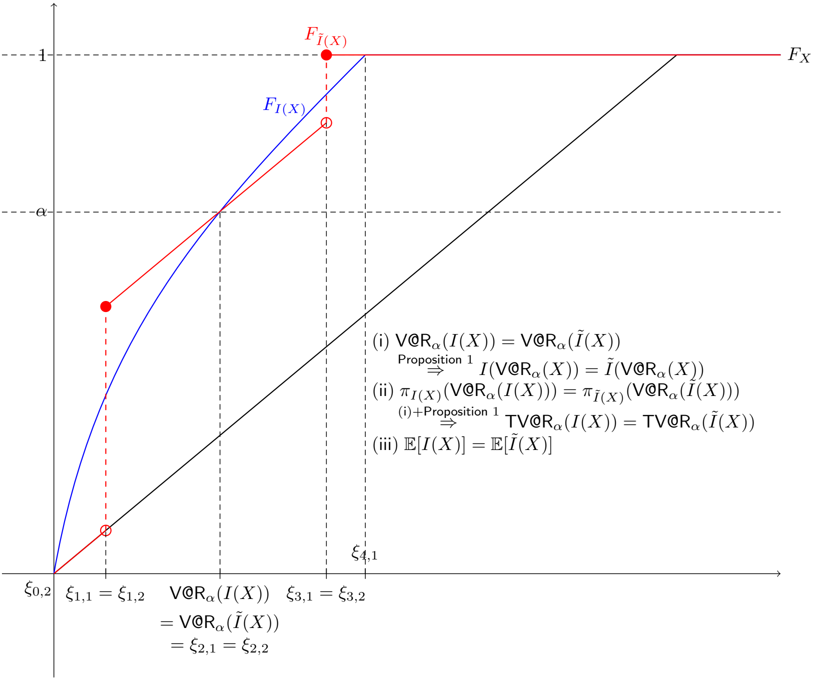

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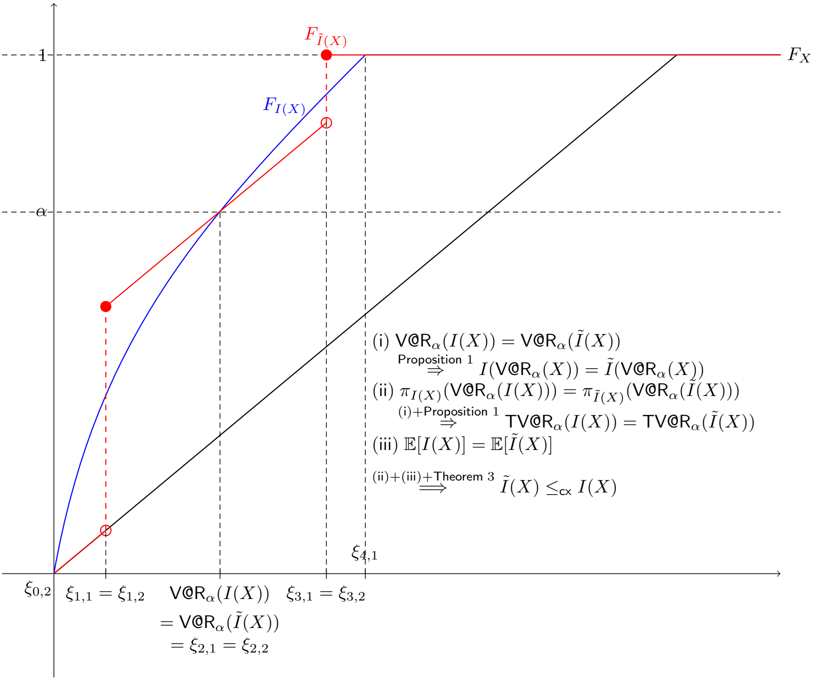
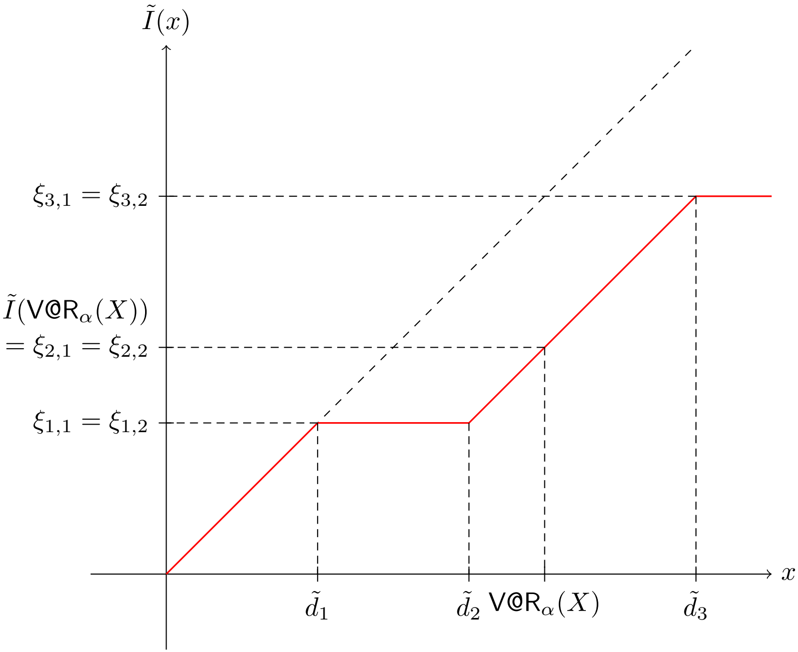
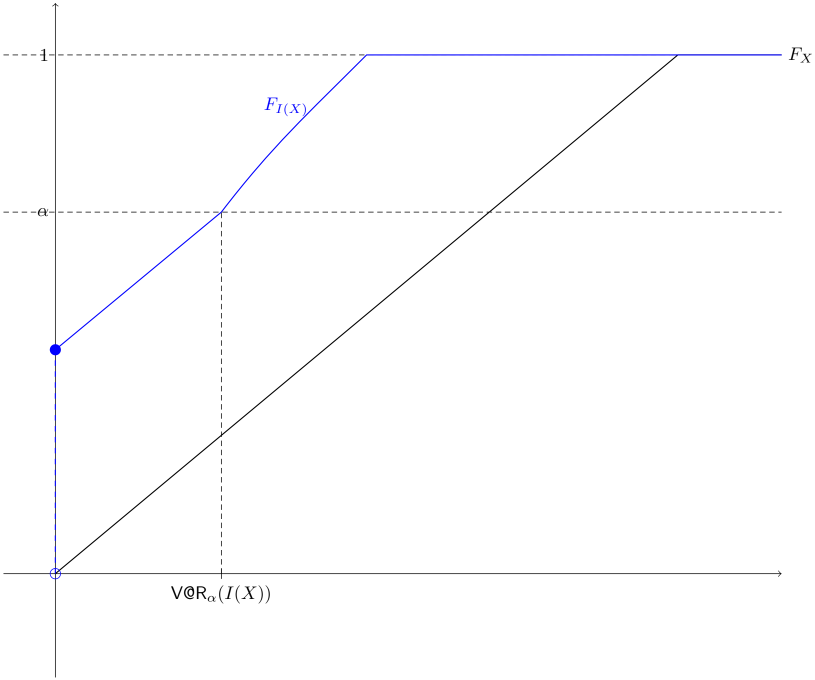
(iii) $\mathbb{E}[I(X)] = \mathbb{E}[\tilde{I}(X)]$
$\xi_{4,1}$
$\xi_{0,2}$
$\xi_{1,1} = \xi_{1,2}$
$\mathsf{V@R}_\alpha(I(X))$
$= \mathsf{V@R}_\alpha(\tilde{I}(X))$
$= \xi_{2,1} = \xi_{2,2}$
$\xi_{3,1} = \xi_{3,2}$

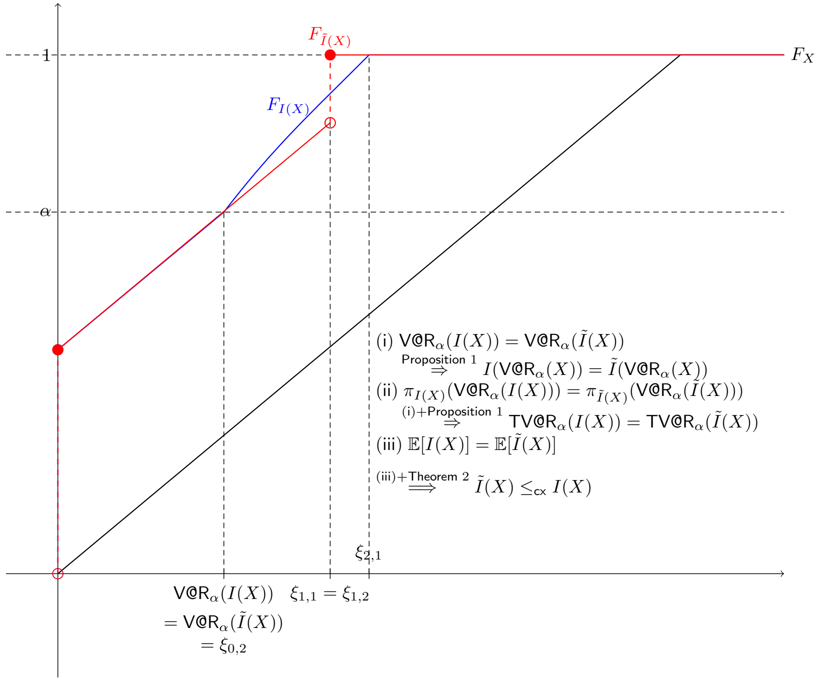


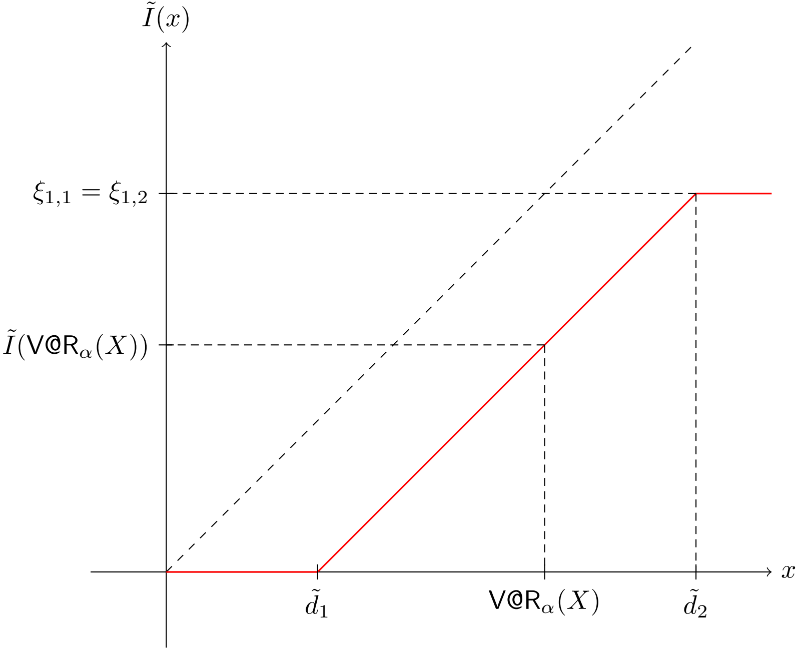
$F_{\tilde{I}(X)}$
1
F_X
$F_{I(X)}$
α
(i) $\mathsf{V@R}_\alpha(I(X)) = \mathsf{V@R}_\alpha(\tilde{I}(X))$
$\overset{\text{Proposition 1}}{\Rightarrow} I(\mathsf{V@R}_\alpha(X)) = \tilde{I}(\mathsf{V@R}_\alpha(X))$
(ii) $\pi_{I(X)}(\mathsf{V@R}_\alpha(I(X))) = \pi_{\tilde{I}(X)}(\mathsf{V@R}_\alpha(\tilde{I}(X)))$
$\overset{\text{(i)+Proposition 1}}{\Rightarrow} \mathsf{TV@R}_\alpha(I(X)) = \mathsf{TV@R}_\alpha(\tilde{I}(X))$
(iii) $\mathbb{E}[I(X)] = \mathbb{E}[\tilde{I}(X)]$
$\overset{\text{(ii)+(iii)+Theorem 3}}{\Longrightarrow} \tilde{I}(X) \leq_{\mathsf{cx}} I(X)$
$\xi_{4,1}$
$\xi_{0,2}$
$\xi_{1,1} = \xi_{1,2}$
$\mathsf{V@R}_\alpha(I(X))$
$= \mathsf{V@R}_\alpha(\tilde{I}(X))$
$= \xi_{2,1} = \xi_{2,2}$
$\xi_{3,1} = \xi_{3,2}$

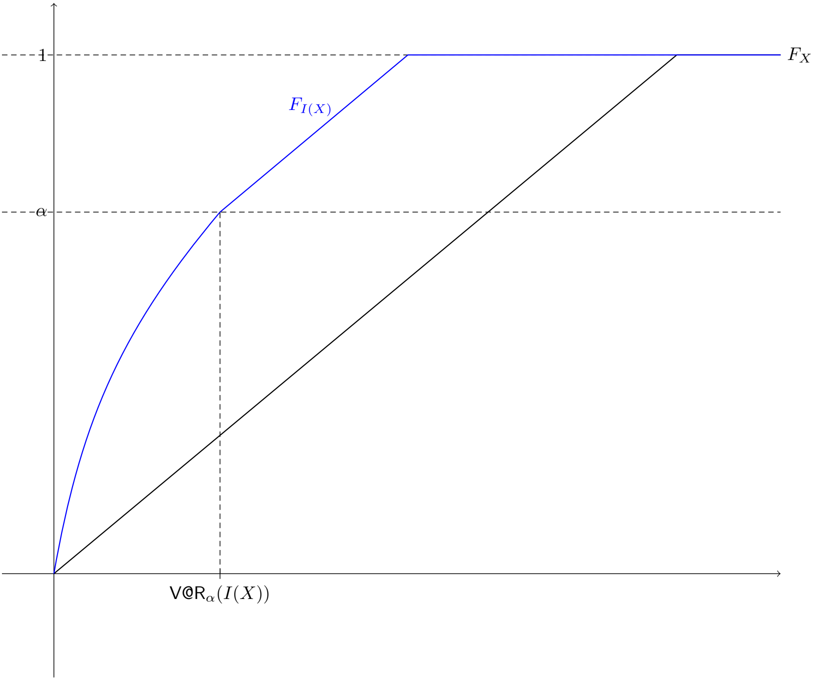

$\tilde{I}(x)$
$\xi_{3,1} = \xi_{3,2}$
$\tilde{I}(\mathsf{V@R}_\alpha(X))$
$= \xi_{2,1} = \xi_{2,2}$
$\xi_{1,1} = \xi_{1,2}$
x
$\tilde{d}_1$
$\tilde{d}_2$
$\mathsf{V@R}_\alpha(X)$
$\tilde{d}_3$

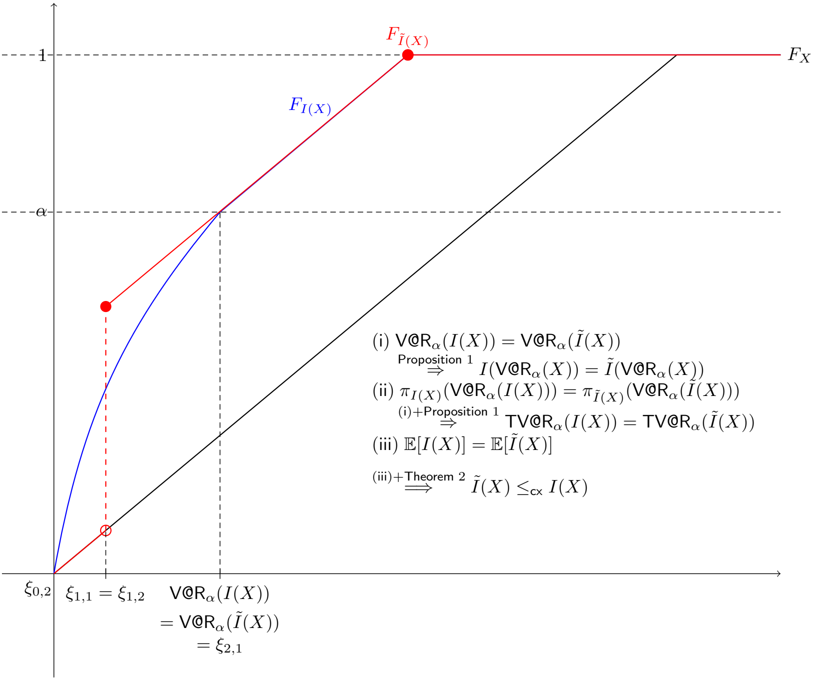


1
α
F_X
$F_{I(X)}$
$\mathsf{V@R}_\alpha(I(X))$

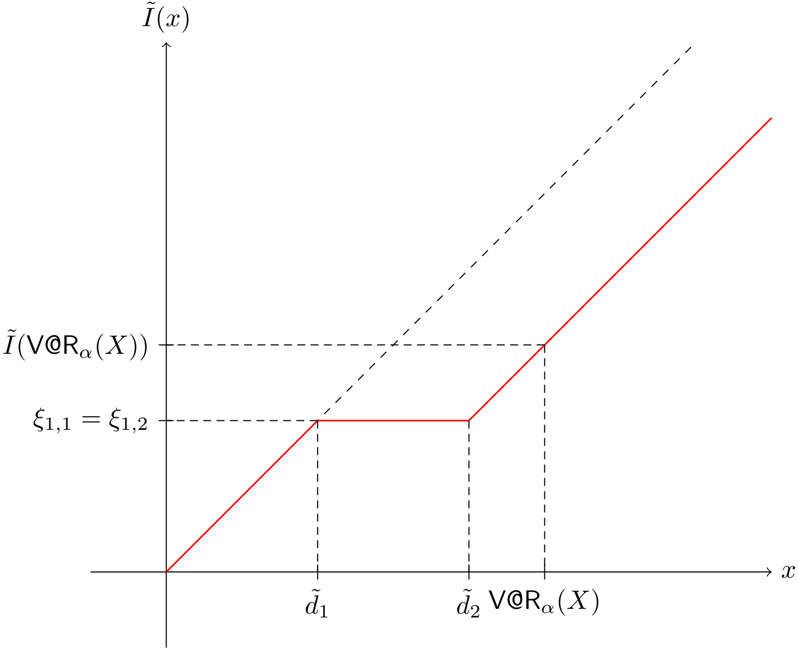

$F_{\tilde{I}(X)}$
1
F_X
$F_{I(X)}$
α
(i) $\mathsf{V@R}_\alpha(I(X)) = \mathsf{V@R}_\alpha(\tilde{I}(X))$
$\overset{\text{Proposition 1}}{\Rightarrow} I(\mathsf{V@R}_\alpha(X)) = \tilde{I}(\mathsf{V@R}_\alpha(X))$
(ii) $\pi_{I(X)}(\mathsf{V@R}_\alpha(I(X))) = \pi_{\tilde{I}(X)}(\mathsf{V@R}_\alpha(\tilde{I}(X)))$
$\overset{\text{(i)+Proposition 1}}{\Rightarrow} \mathsf{TV@R}_\alpha(I(X)) = \mathsf{TV@R}_\alpha(\tilde{I}(X))$
(iii) $\mathbb{E}[I(X)] = \mathbb{E}[\tilde{I}(X)]$
$\overset{\text{(iii)+Theorem 2}}{\Longrightarrow} \tilde{I}(X) \leq_{\mathsf{cx}} I(X)$
$\xi_{2,1}$
$\mathsf{V@R}_\alpha(I(X))$
$= \mathsf{V@R}_\alpha(\tilde{I}(X))$
$= \xi_{0,2}$
$\xi_{1,1} = \xi_{1,2}$


$\tilde{I}(x)$
$\xi_{1,1} = \xi_{1,2}$
$\tilde{I}(\mathsf{V@R}_\alpha(X))$
x
$\tilde{d}_1$
$\mathsf{V@R}_\alpha(X)$
$\tilde{d}_2$


1
α
F_X
$F_{I(X)}$
$\mathsf{V@R}_\alpha(I(X))$



$F_{\tilde{I}(X)}$
1
F_X
$F_{I(X)}$
α
(i) $\mathsf{V@R}_\alpha(I(X)) = \mathsf{V@R}_\alpha(\tilde{I}(X))$
$\overset{\text{Proposition 1}}{\Rightarrow} I(\mathsf{V@R}_\alpha(X)) = \tilde{I}(\mathsf{V@R}_\alpha(X))$
(ii) $\pi_{I(X)}(\mathsf{V@R}_\alpha(I(X))) = \pi_{\tilde{I}(X)}(\mathsf{V@R}_\alpha(\tilde{I}(X)))$
$\overset{\text{(i)+Proposition 1}}{\Rightarrow} \mathsf{TV@R}_\alpha(I(X)) = \mathsf{TV@R}_\alpha(\tilde{I}(X))$
(iii) $\mathbb{E}[I(X)] = \mathbb{E}[\tilde{I}(X)]$
$\overset{\text{(iii)+Theorem 2}}{\Longrightarrow} \tilde{I}(X) \leq_{\mathsf{cx}} I(X)$
$\xi_{0,2}$
$\xi_{1,1} = \xi_{1,2}$
$\mathsf{V@R}_\alpha(I(X))$
$= \mathsf{V@R}_\alpha(\tilde{I}(X))$
$= \xi_{2,1}$



$\tilde{I}(x)$
$\tilde{I}(\mathsf{V@R}_\alpha(X))$
$\xi_{1,1} = \xi_{1,2}$
x
$\tilde{d}_1$
$\tilde{d}_2$
$\mathsf{V@R}_\alpha(X)$

Solution

To explicitly clarify the dependence of G on its three arguments, define the indicators:

$$J_i\left(G\right) \triangleq \begin{cases} 0 & \text{if } G \text{ does not depend on the } i\text{-th argument,} \\ 1 & \text{otherwise,} \end{cases}$$

for $i = 1, 2, 3$.

Theorem 4

Assume that G depends on all three arguments, such that $J_1(G) = J_2(G) = J_3(G) = 1$. The optimal ceded loss function for the problem takes the form of

$$I^*(x) = x - (x - d_1^*)_+ + (x - d_2^*)_+ - (x - d_3^*)_+,$$

for some $0 \leq d_1^* \leq d_2^* \leq V@R_\alpha(X) \leq d_3^* \leq \operatorname{ess\,sup} X$.

Solution (cont.)

The construction of $\tilde{I}$ such that

$$\mathsf{TV@R}_\alpha(I(X)) = \mathsf{TV@R}_\alpha(\tilde{I}(X))$$

is due to both

$$\mathsf{V@R}_\alpha(I(X)) = \mathsf{V@R}_\alpha(\tilde{I}(X)) \tag{1}$$

and, if $\alpha < 1$,

$$\pi_{I(X)}(\mathsf{V@R}_\alpha(I(X))) = \pi_{\tilde{I}(X)}(\mathsf{V@R}_\alpha(\tilde{I}(X))). \tag{2}$$

Therefore, even if G does not depend on V@R but depends on TV@R, (1) is still needed to be held.

Theorem 5 (Extension of Chi (2012))

Assume that G depends only on the first and the third arguments, such that $J_1(G) = J_3(G) = 1$ but $J_2(G) = 0$. The optimal ceded loss function for the problem takes the form of

$$I^*(x) = x - (x - d_1^*)_+ + (x - d_2^*)_+ - (x - d_3^*)_+,$$

for some $0 \le d_1^* \le d_2^* \le \mathsf{V@R}_\alpha(X) \le d_3^* \le \operatorname{ess\,sup} X$.

Solution (cont.)

If G does not depend on TV@R but depends on V@R, (2) is not necessary anymore.

Theorem 6 (Extension of Chi (2012))

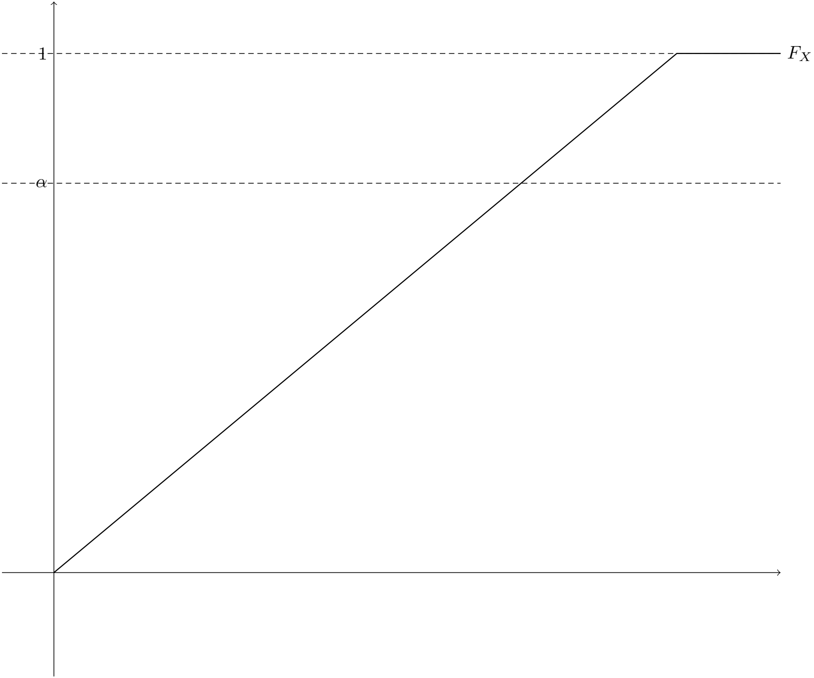
Assume that G depends only on the first and the second arguments, such that $J_1(G) = J_2(G) = 1$ but $J_3(G) = 0$. The optimal ceded loss function for the problem takes the form of either

$$I^*(x) = x - (x - d_1^*)_+ + (x - d_2^*)_+ - (x - \text{V@R}_\alpha(X))_+,$$

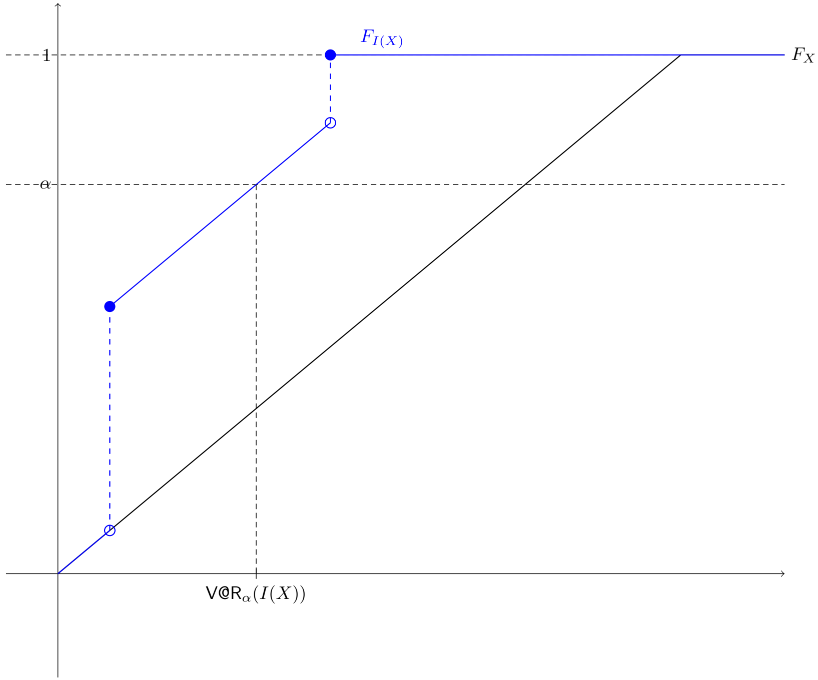
for some $0 \leq d_1^* \leq d_2^* \leq \text{V@R}_\alpha(X) \leq \operatorname{ess\,sup} X$, *or*

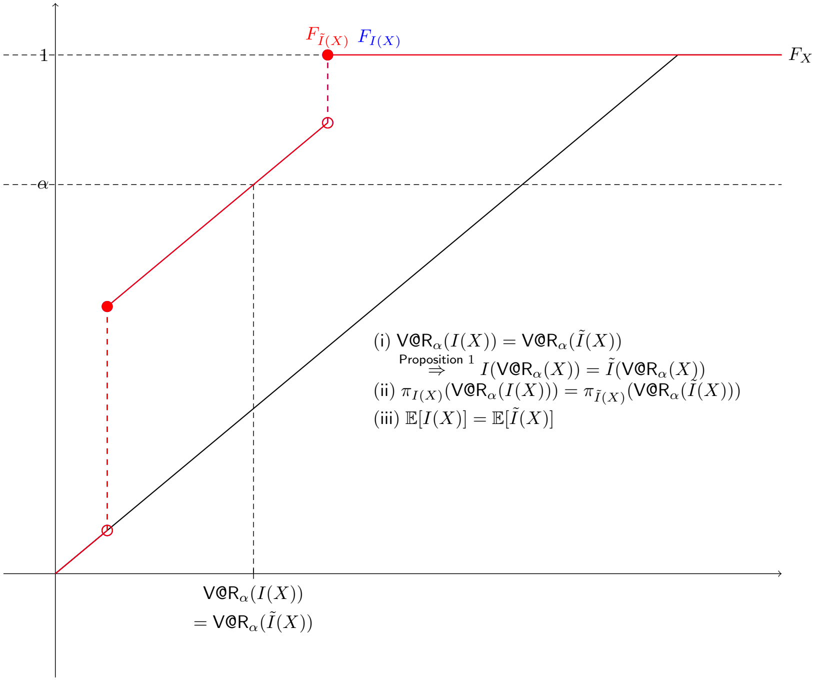
$$I^{**}(x) = x - (x - d^{**})_+,$$

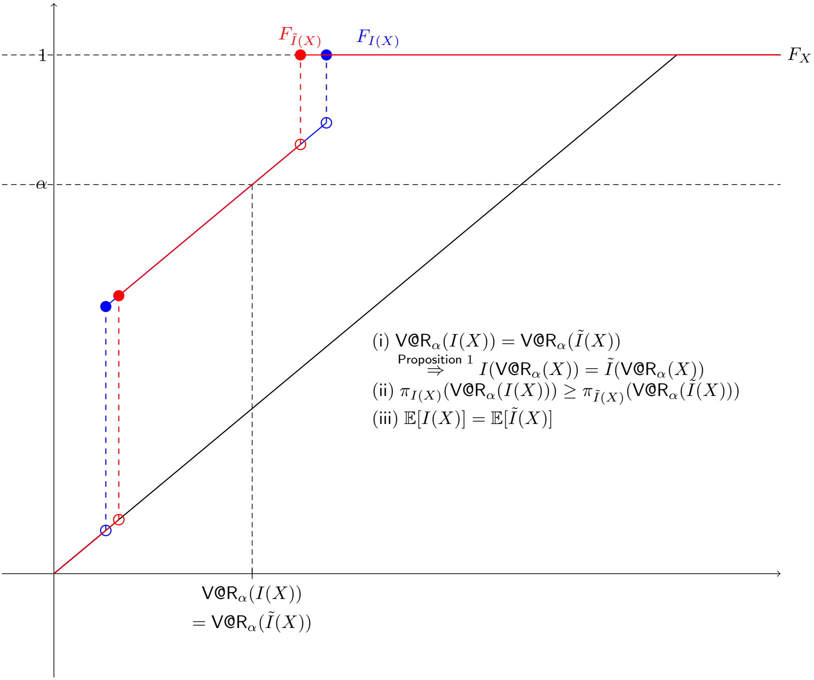
for some $0 \leq d^{**} \leq \operatorname{ess\,sup} X$.

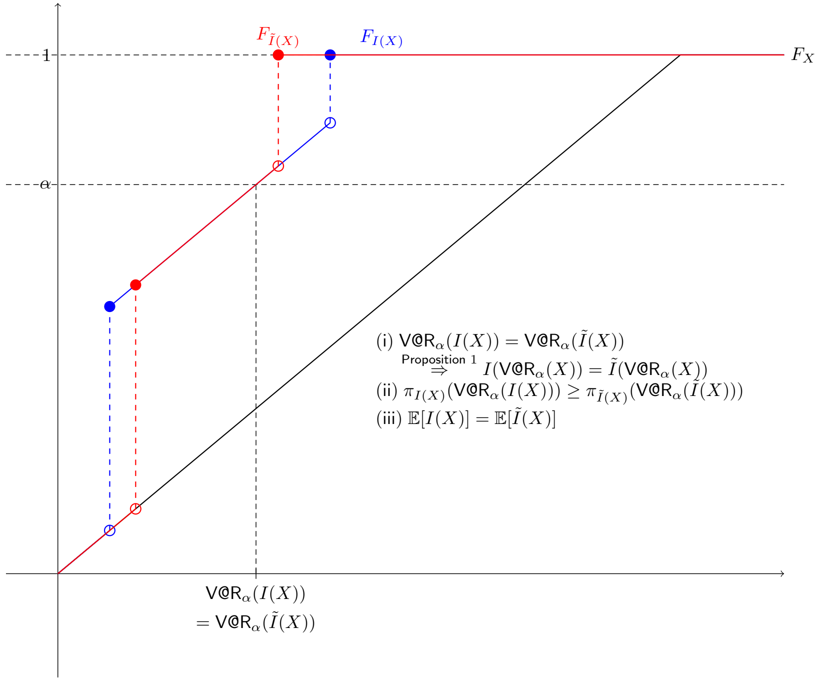


1
α
F_X


$F_{I(X)}$
F_X
1
α
$\mathsf{V@R}_\alpha(I(X))$


$F_{\tilde{I}(X)}$
$F_{I(X)}$
F_X
1
α
(i) $\mathsf{V@R}_\alpha(I(X)) = \mathsf{V@R}_\alpha(\tilde{I}(X))$
$\overset{\text{Proposition 1}}{\Rightarrow} I(\mathsf{V@R}_\alpha(X)) = \tilde{I}(\mathsf{V@R}_\alpha(X))$
(ii) $\pi_{I(X)}(\mathsf{V@R}_\alpha(I(X))) = \pi_{\tilde{I}(X)}(\mathsf{V@R}_\alpha(\tilde{I}(X)))$
(iii) $\mathbb{E}[I(X)] = \mathbb{E}[\tilde{I}(X)]$
$\mathsf{V@R}_\alpha(I(X))$
$= \mathsf{V@R}_\alpha(\tilde{I}(X))$


$F_{\tilde{I}(X)}$
$F_{I(X)}$
1
F_X
α
(i) $\mathsf{V@R}_\alpha(I(X)) = \mathsf{V@R}_\alpha(\tilde{I}(X))$
$\overset{\text{Proposition 1}}{\Rightarrow} I(\mathsf{V@R}_\alpha(X)) = \tilde{I}(\mathsf{V@R}_\alpha(X))$
(ii) $\pi_{I(X)}(\mathsf{V@R}_\alpha(I(X))) \geq \pi_{\tilde{I}(X)}(\mathsf{V@R}_\alpha(\tilde{I}(X)))$
(iii) $\mathbb{E}[I(X)] = \mathbb{E}[\tilde{I}(X)]$
$\mathsf{V@R}_\alpha(I(X))$
$= \mathsf{V@R}_\alpha(\tilde{I}(X))$



$F_{\tilde{I}(X)}$
$F_{I(X)}$
F_X
1
α
(i) $\mathsf{V@R}_\alpha(I(X)) = \mathsf{V@R}_\alpha(\tilde{I}(X))$
$\overset{\text{Proposition 1}}{\Rightarrow} I(\mathsf{V@R}_\alpha(X)) = \tilde{I}(\mathsf{V@R}_\alpha(X))$
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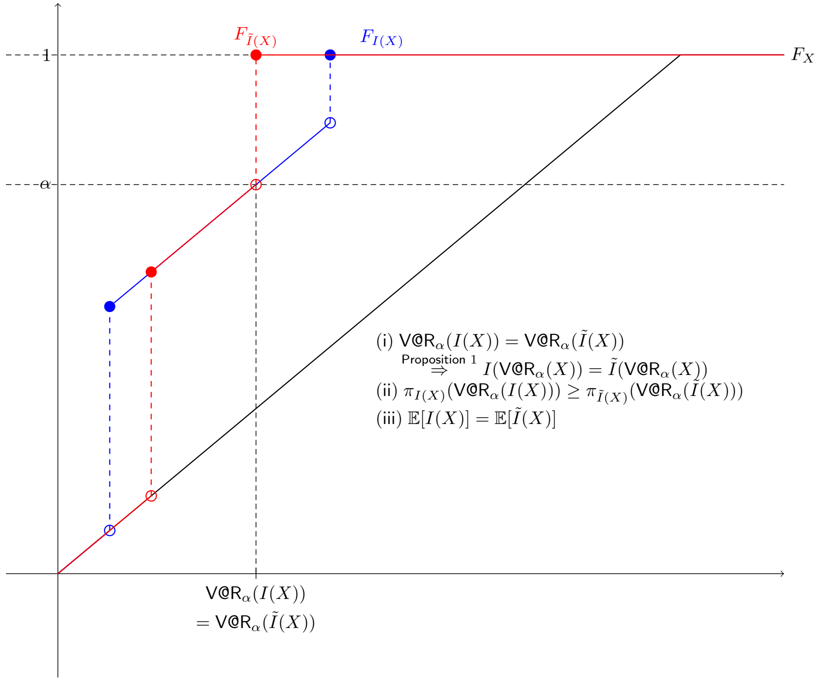
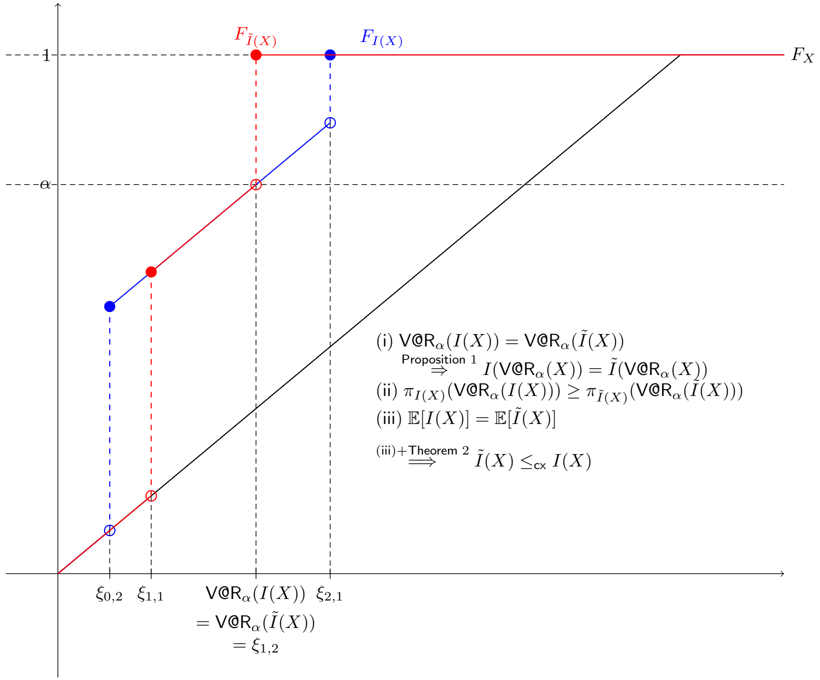
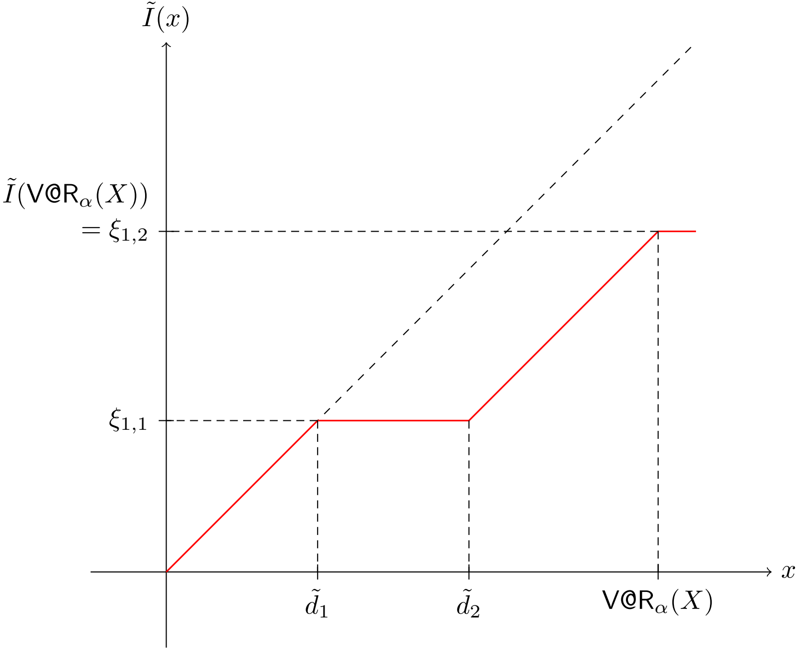
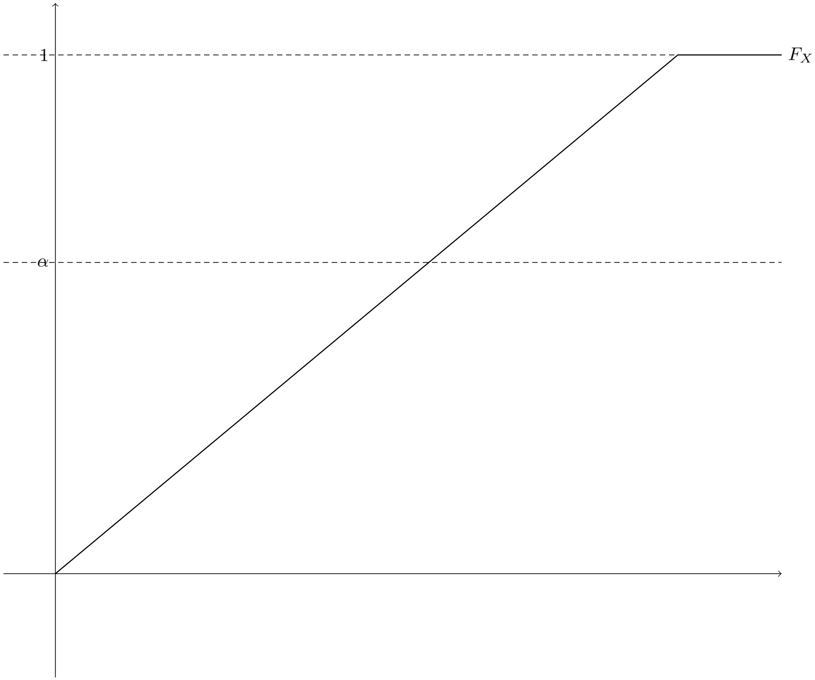
$\mathsf{V@R}_\alpha(I(X))$
$= \mathsf{V@R}_\alpha(\tilde{I}(X))$

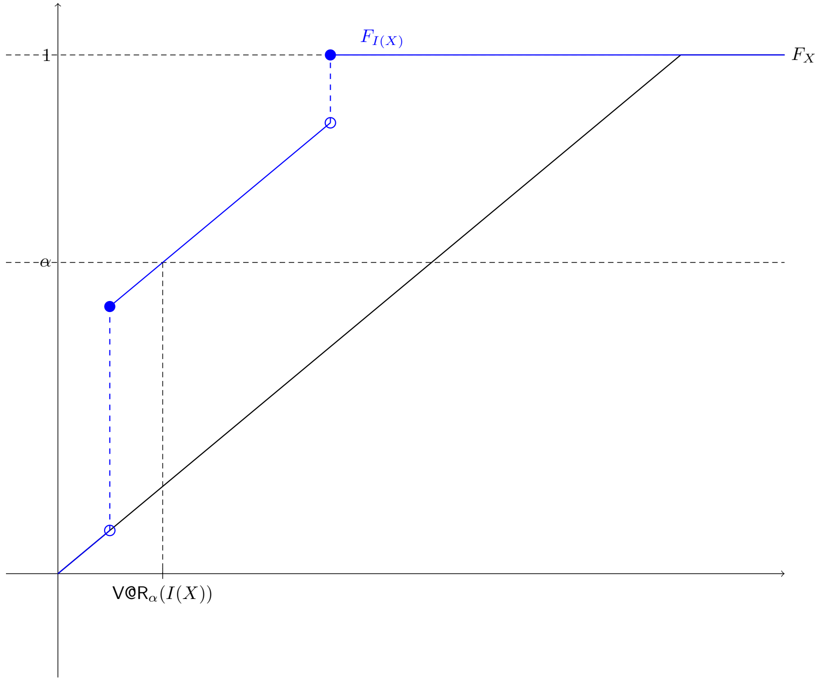

$F_{\tilde{I}(X)}$
$F_{I(X)}$
1
F_X
α
(i) $\mathsf{V@R}_\alpha(I(X)) = \mathsf{V@R}_\alpha(\tilde{I}(X))$
$\overset{\text{Proposition 1}}{\Rightarrow} I(\mathsf{V@R}_\alpha(X)) = \tilde{I}(\mathsf{V@R}_\alpha(X))$
(ii) $\pi_{I(X)}(\mathsf{V@R}_\alpha(I(X))) \geq \pi_{\tilde{I}(X)}(\mathsf{V@R}_\alpha(\tilde{I}(X)))$
(iii) $\mathbb{E}[I(X)] = \mathbb{E}[\tilde{I}(X)]$
$\mathsf{V@R}_\alpha(I(X))$
$= \mathsf{V@R}_\alpha(\tilde{I}(X))$

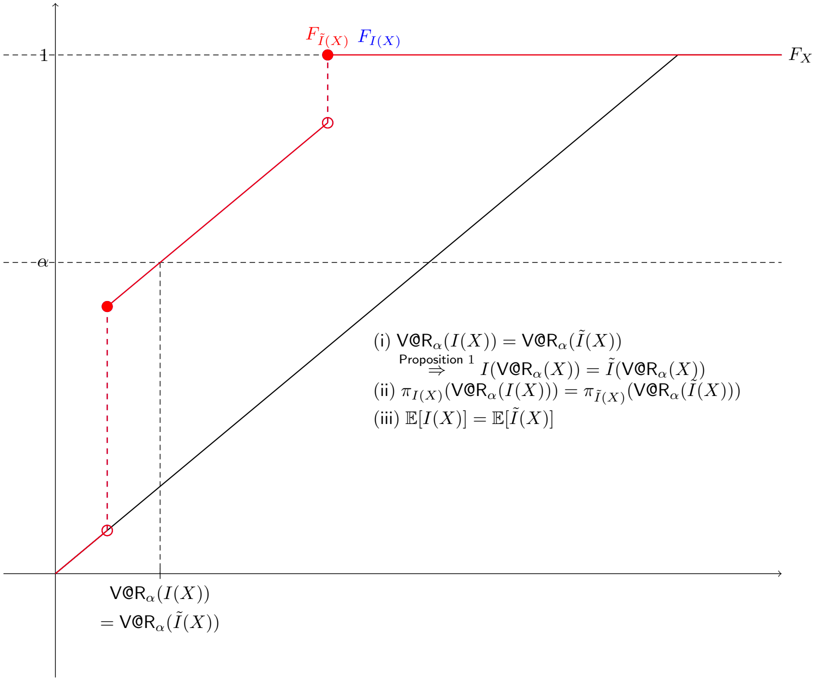


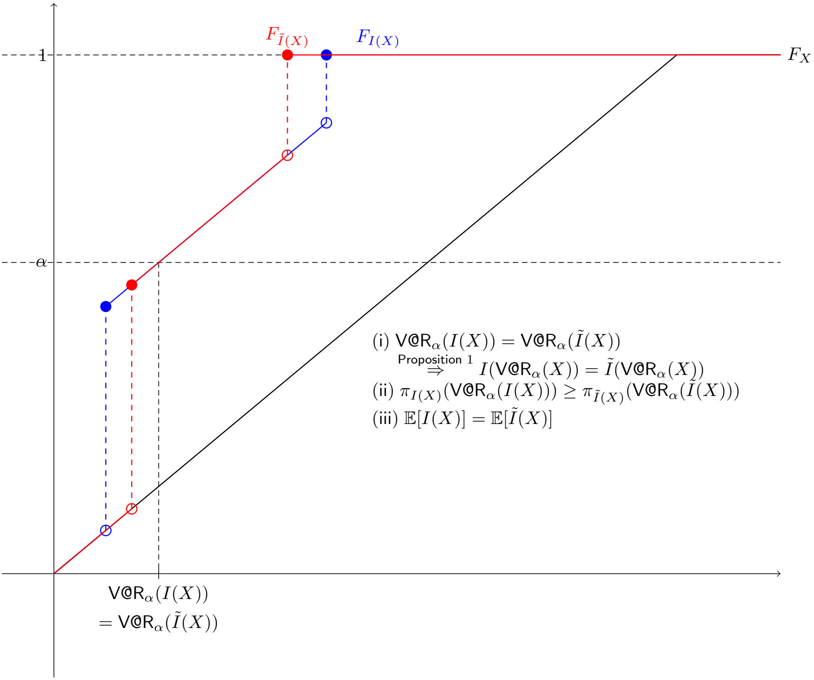
$F_{\tilde{I}(X)}$
$F_{I(X)}$
F_X
1
α
(i) $\mathsf{V@R}_\alpha(I(X)) = \mathsf{V@R}_\alpha(\tilde{I}(X))$
$\overset{\text{Proposition 1}}{\Rightarrow} I(\mathsf{V@R}_\alpha(X)) = \tilde{I}(\mathsf{V@R}_\alpha(X))$
(ii) $\pi_{I(X)}(\mathsf{V@R}_\alpha(I(X))) \geq \pi_{\tilde{I}(X)}(\mathsf{V@R}_\alpha(\tilde{I}(X)))$
(iii) $\mathbb{E}[I(X)] = \mathbb{E}[\tilde{I}(X)]$
$\overset{\text{(iii)+Theorem 2}}{\Longrightarrow} \tilde{I}(X) \leq_{\mathsf{cx}} I(X)$
$\xi_{0,2}$
$\xi_{1,1}$
$\mathsf{V@R}_\alpha(I(X))$
$= \mathsf{V@R}_\alpha(\tilde{I}(X))$
$= \xi_{1,2}$
$\xi_{2,1}$


$\tilde{I}(x)$
$\tilde{I}(\mathsf{V@R}_\alpha(X))$
$= \xi_{1,2}$
$\xi_{1,1}$
x
$\tilde{d}_1$
$\tilde{d}_2$
$\mathsf{V@R}_\alpha(X)$

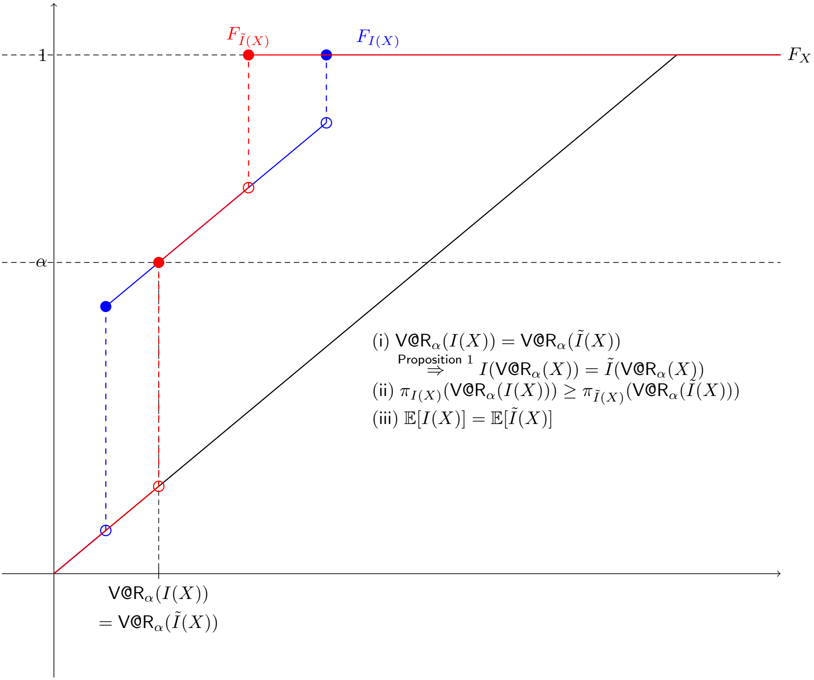

1
α
F_X


$F_{I(X)}$
F_X
1
α
$\mathsf{V@R}_\alpha(I(X))$


$F_{\tilde{I}(X)}$ $F_{I(X)}$
F_X
1
α
(i) $\mathsf{V@R}_\alpha(I(X)) = \mathsf{V@R}_\alpha(\tilde{I}(X))$
$\overset{\text{Proposition 1}}{\Rightarrow} I(\mathsf{V@R}_\alpha(X)) = \tilde{I}(\mathsf{V@R}_\alpha(X))$
(ii) $\pi_{I(X)}(\mathsf{V@R}_\alpha(I(X))) = \pi_{\tilde{I}(X)}(\mathsf{V@R}_\alpha(\tilde{I}(X)))$
(iii) $\mathbb{E}[I(X)] = \mathbb{E}[\tilde{I}(X)]$
$\mathsf{V@R}_\alpha(I(X))$
$= \mathsf{V@R}_\alpha(\tilde{I}(X))$



$F_{\tilde{I}(X)}$
$F_{I(X)}$
1
F_X
α
(i) $\mathsf{V@R}_\alpha(I(X)) = \mathsf{V@R}_\alpha(\tilde{I}(X))$
$\overset{\text{Proposition 1}}{\Rightarrow} I(\mathsf{V@R}_\alpha(X)) = \tilde{I}(\mathsf{V@R}_\alpha(X))$
(ii) $\pi_{I(X)}(\mathsf{V@R}_\alpha(I(X))) \geq \pi_{\tilde{I}(X)}(\mathsf{V@R}_\alpha(\tilde{I}(X)))$
(iii) $\mathbb{E}[I(X)] = \mathbb{E}[\tilde{I}(X)]$
$\mathsf{V@R}_\alpha(I(X))$
$= \mathsf{V@R}_\alpha(\tilde{I}(X))$



$F_{\tilde{I}(X)}$
$F_{I(X)}$
1
F_X
α
(i) $\mathsf{V@R}_\alpha(I(X)) = \mathsf{V@R}_\alpha(\tilde{I}(X))$
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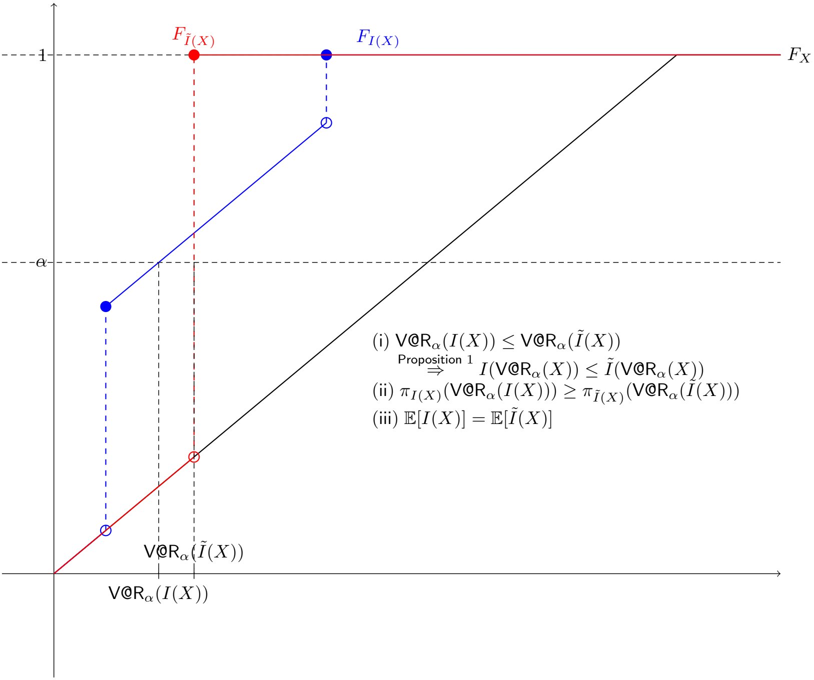
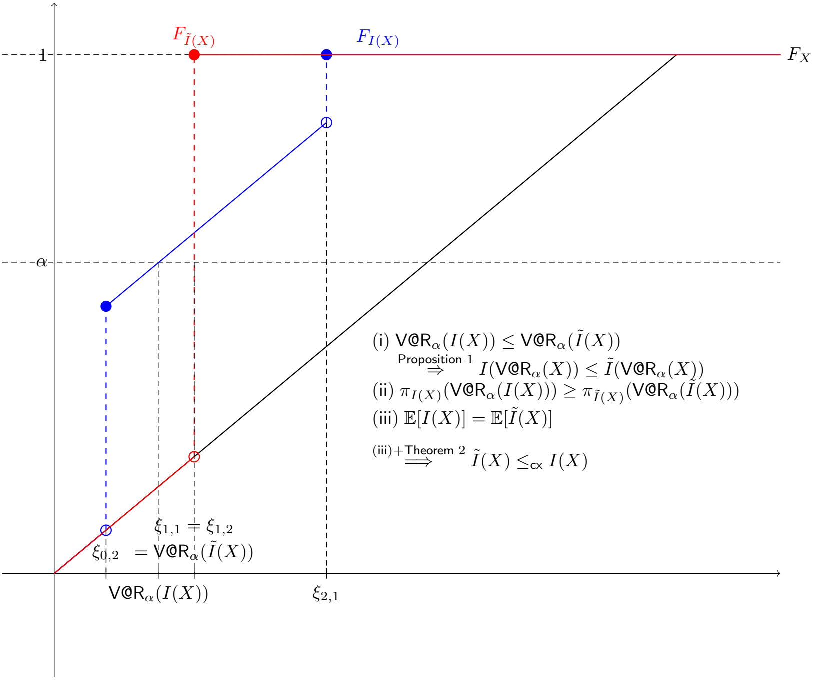
(ii) $\pi_{I(X)}(\mathsf{V@R}_\alpha(I(X))) \geq \pi_{\tilde{I}(X)}(\mathsf{V@R}_\alpha(\tilde{I}(X)))$
(iii) $\mathbb{E}[I(X)] = \mathbb{E}[\tilde{I}(X)]$
$\mathsf{V@R}_\alpha(I(X))$
$= \mathsf{V@R}_\alpha(\tilde{I}(X))$


$F_{\tilde{I}(X)}$
$F_{I(X)}$
F_X
1
α
(i) $\text{V@R}_\alpha(I(X)) \leq \text{V@R}_\alpha(\tilde{I}(X))$
$\overset{\text{Proposition 1}}{\Rightarrow} I(\text{V@R}_\alpha(X)) \leq \tilde{I}(\text{V@R}_\alpha(X))$
(ii) $\pi_{I(X)}(\text{V@R}_\alpha(I(X))) \geq \pi_{\tilde{I}(X)}(\text{V@R}_\alpha(\tilde{I}(X)))$
(iii) $\mathbb{E}[I(X)] = \mathbb{E}[\tilde{I}(X)]$
$\text{V@R}_\alpha(\tilde{I}(X))$
$\text{V@R}_\alpha(I(X))$


$F_{\tilde{I}(X)}$
$F_{I(X)}$
F_X
1
α
(i) $\mathsf{V@R}_\alpha(I(X)) \le \mathsf{V@R}_\alpha(\tilde{I}(X))$
$\overset{\text{Proposition 1}}{\Rightarrow} I(\mathsf{V@R}_\alpha(X)) \le \tilde{I}(\mathsf{V@R}_\alpha(X))$
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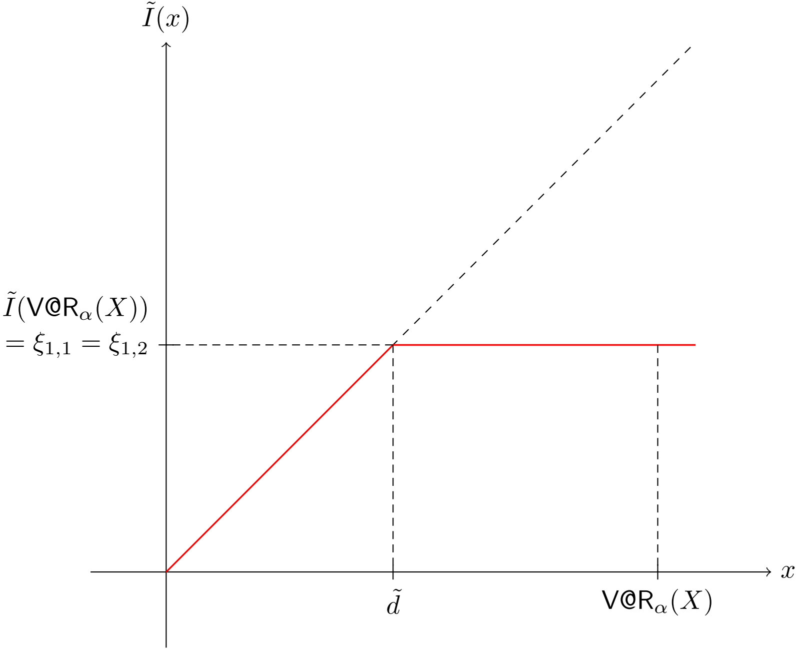
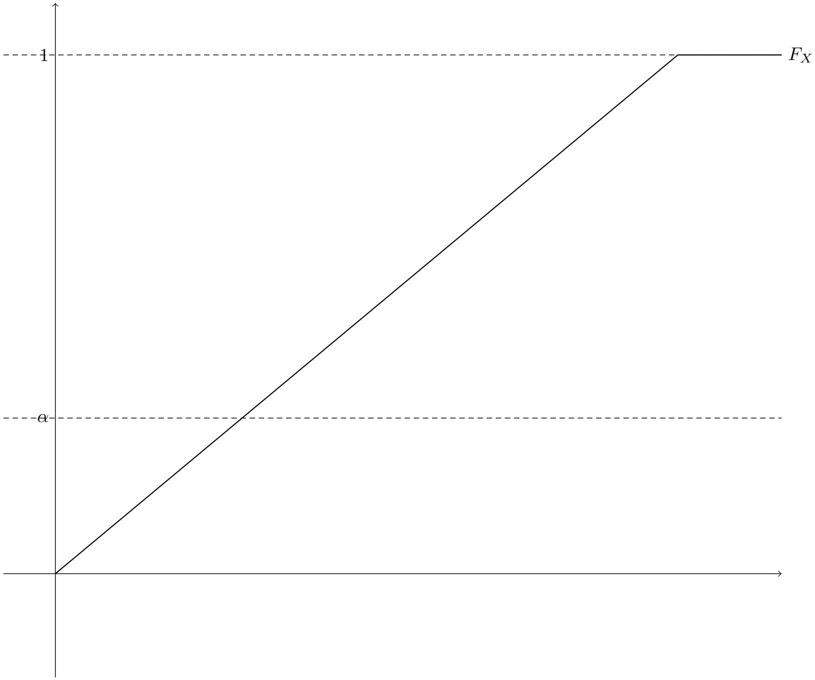
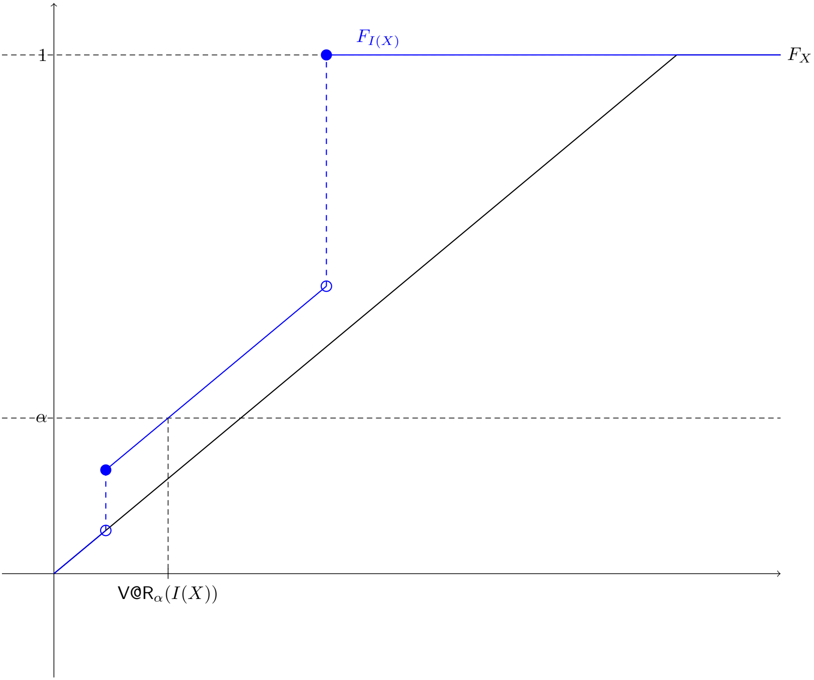
$\overset{\text{(iii)+Theorem 2}}{\Longrightarrow} \tilde{I}(X) \le_{\mathsf{cx}} I(X)$
$\xi_{1,1} = \xi_{1,2}$
$\xi_{0,2}$
$= \mathsf{V@R}_\alpha(\tilde{I}(X))$
$\mathsf{V@R}_\alpha(I(X))$
$\xi_{2,1}$

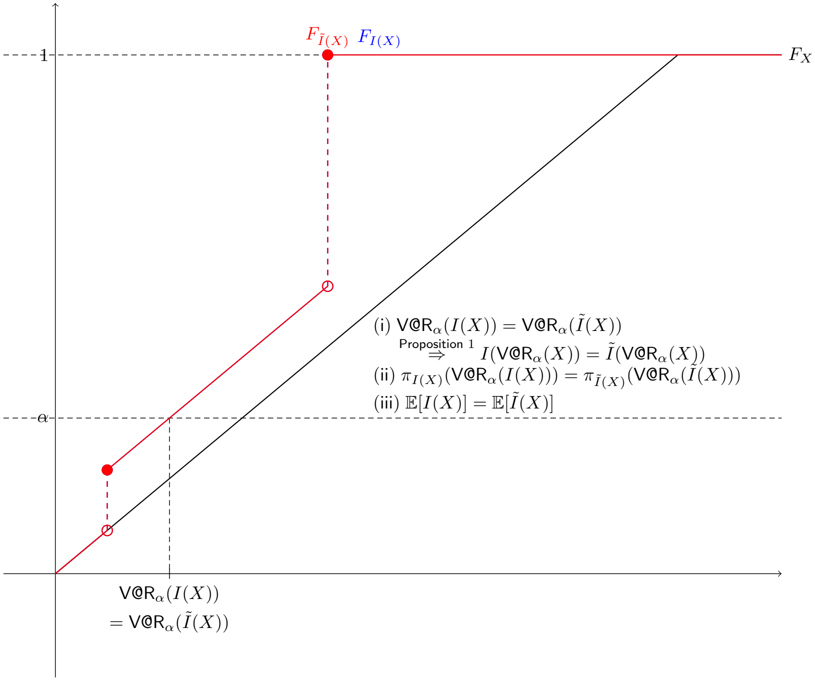


$\tilde{I}(x)$
$\tilde{I}(\mathsf{V@R}_\alpha(X))$
$= \xi_{1,1} = \xi_{1,2}$
x
$\tilde{d}$
$\mathsf{V@R}_\alpha(X)$

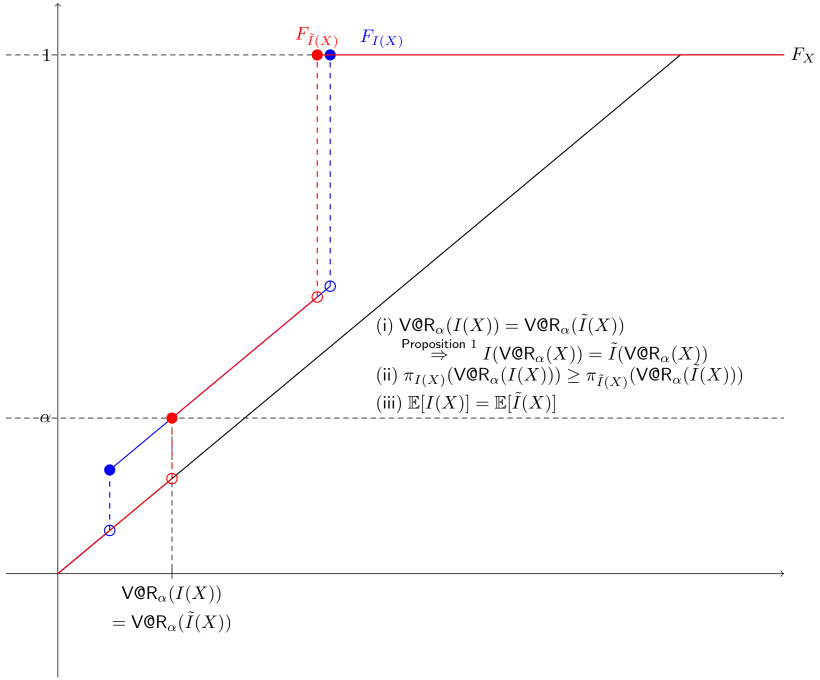

1
α
F_X



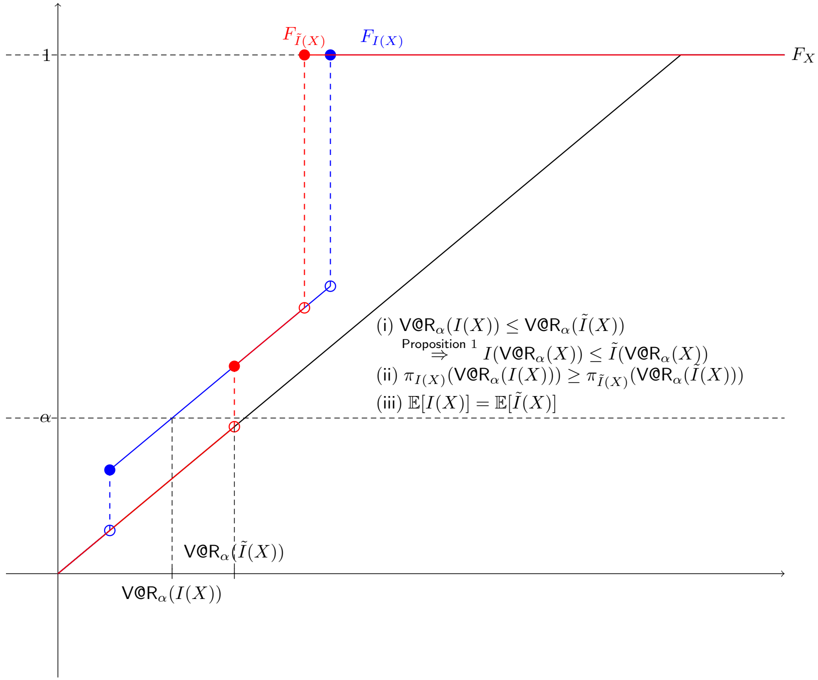
$F_{I(X)}$
1
F_X
α
$\mathsf{V@R}_\alpha(I(X))$

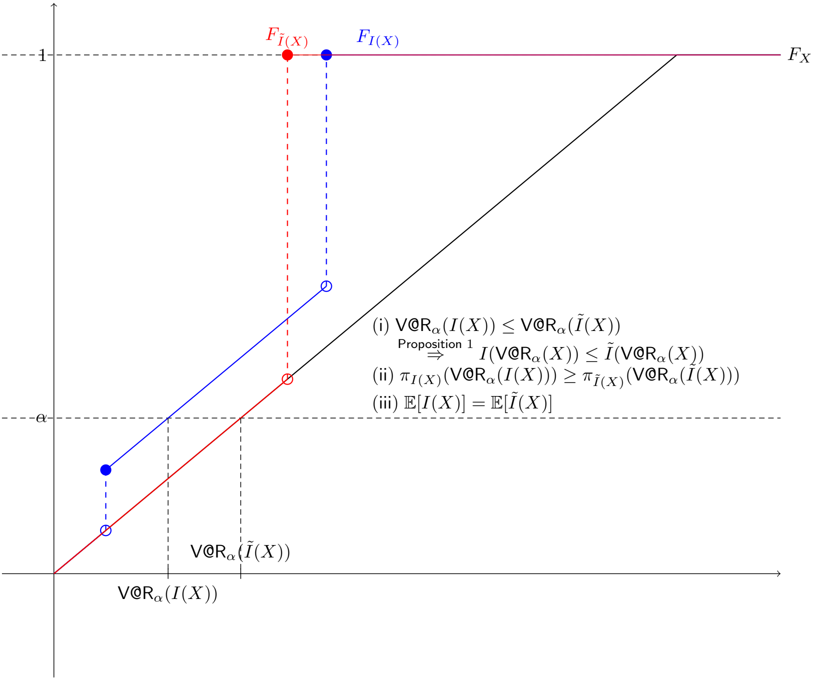


$F_{\tilde{I}(X)}$ $F_{I(X)}$
1
F_X
(i) $\mathsf{V@R}_\alpha(I(X)) = \mathsf{V@R}_\alpha(\tilde{I}(X))$
$\overset{\text{Proposition 1}}{\Rightarrow} I(\mathsf{V@R}_\alpha(X)) = \tilde{I}(\mathsf{V@R}_\alpha(X))$
(ii) $\pi_{I(X)}(\mathsf{V@R}_\alpha(I(X))) = \pi_{\tilde{I}(X)}(\mathsf{V@R}_\alpha(\tilde{I}(X)))$
(iii) $\mathbb{E}[I(X)] = \mathbb{E}[\tilde{I}(X)]$
α
$\mathsf{V@R}_\alpha(I(X))$
$= \mathsf{V@R}_\alpha(\tilde{I}(X))$

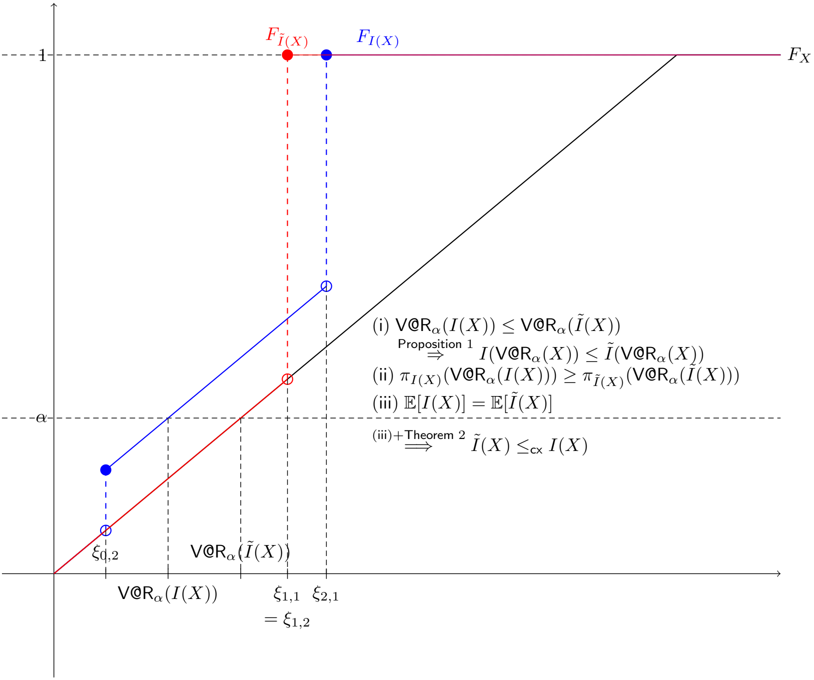


$F_{\tilde{I}(X)}$
$F_{I(X)}$
1
F_X
(i) $\mathsf{V@R}_\alpha(I(X)) = \mathsf{V@R}_\alpha(\tilde{I}(X))$
$\overset{\text{Proposition 1}}{\Rightarrow} I(\mathsf{V@R}_\alpha(X)) = \tilde{I}(\mathsf{V@R}_\alpha(X))$
(ii) $\pi_{I(X)}(\mathsf{V@R}_\alpha(I(X))) \geq \pi_{\tilde{I}(X)}(\mathsf{V@R}_\alpha(\tilde{I}(X)))$
(iii) $\mathbb{E}[I(X)] = \mathbb{E}[\tilde{I}(X)]$
α
$\mathsf{V@R}_\alpha(I(X))$
$= \mathsf{V@R}_\alpha(\tilde{I}(X))$

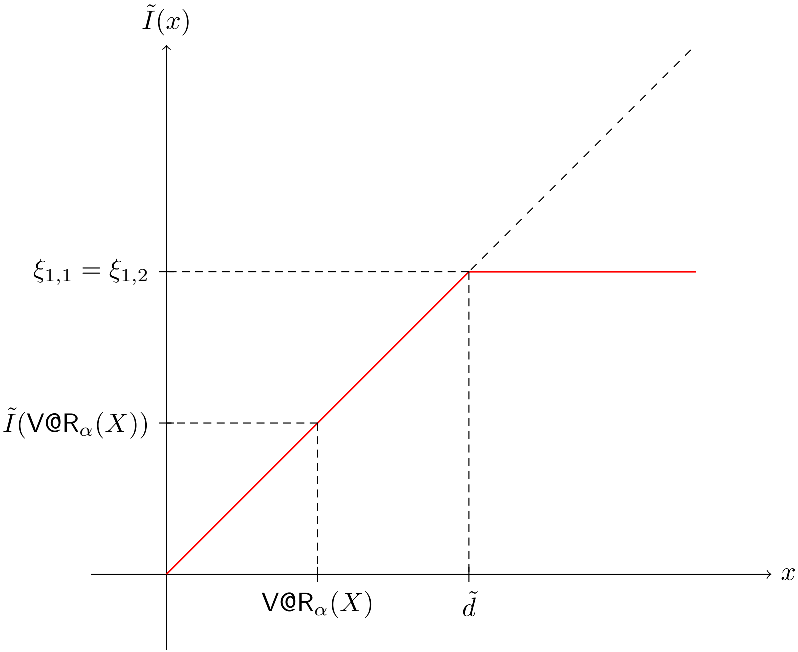

$F_{\tilde{I}(X)}$
$F_{I(X)}$
1
F_X
(i) V@R$_\alpha(I(X)) \leq$ V@R$_\alpha(\tilde{I}(X))$
$\overset{\text{Proposition 1}}{\Rightarrow}$ I(V@R$_\alpha(X)) \leq \tilde{I}$(V@R$_\alpha(X))$
(ii) $\pi_{I(X)}$(V@R$_\alpha(I(X))) \geq \pi_{\tilde{I}(X)}$(V@R$_\alpha(\tilde{I}(X)))$
(iii) $\mathbb{E}[I(X)] = \mathbb{E}[\tilde{I}(X)]$
α
V@R$_\alpha(\tilde{I}(X))$
V@R$_\alpha(I(X))$



$F_{\tilde{I}(X)}$
$F_{I(X)}$
1
F_X
(i) $\mathsf{V@R}_\alpha(I(X)) \le \mathsf{V@R}_\alpha(\tilde{I}(X))$
$\overset{\text{Proposition 1}}{\Rightarrow}$ $I(\mathsf{V@R}_\alpha(X)) \le \tilde{I}(\mathsf{V@R}_\alpha(X))$
(ii) $\pi_{I(X)}(\mathsf{V@R}_\alpha(I(X))) \ge \pi_{\tilde{I}(X)}(\mathsf{V@R}_\alpha(\tilde{I}(X)))$
(iii) $\mathbb{E}[I(X)] = \mathbb{E}[\tilde{I}(X)]$
α
$\mathsf{V@R}_\alpha(\tilde{I}(X))$
$\mathsf{V@R}_\alpha(I(X))$



$F_{\tilde{I}(X)}$
$F_{I(X)}$
F_X
1
α
(i) $\mathsf{V@R}_\alpha(I(X)) \leq \mathsf{V@R}_\alpha(\tilde{I}(X))$
$\overset{\text{Proposition 1}}{\Rightarrow} I(\mathsf{V@R}_\alpha(X)) \leq \tilde{I}(\mathsf{V@R}_\alpha(X))$
(ii) $\pi_{I(X)}(\mathsf{V@R}_\alpha(I(X))) \geq \pi_{\tilde{I}(X)}(\mathsf{V@R}_\alpha(\tilde{I}(X)))$
(iii) $\mathbb{E}[I(X)] = \mathbb{E}[\tilde{I}(X)]$
$\overset{\text{(iii)+Theorem 2}}{\Longrightarrow} \tilde{I}(X) \leq_{\mathsf{cx}} I(X)$
$\xi_{0,2}$
$\mathsf{V@R}_\alpha(\tilde{I}(X))$
$\mathsf{V@R}_\alpha(I(X))$
$\xi_{1,1}$
$= \xi_{1,2}$
$\xi_{2,1}$


$\tilde{I}(x)$
$\xi_{1,1} = \xi_{1,2}$
$\tilde{I}(\mathsf{V@R}_\alpha(X))$
x
$\mathsf{V@R}_\alpha(X)$
$\tilde{d}$

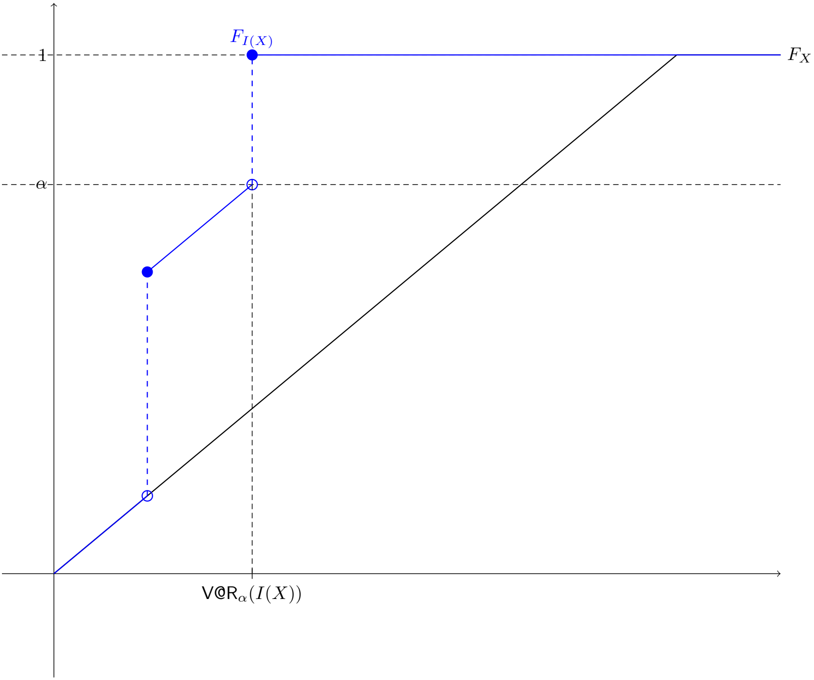
Solution (cont.)

Theorem 7

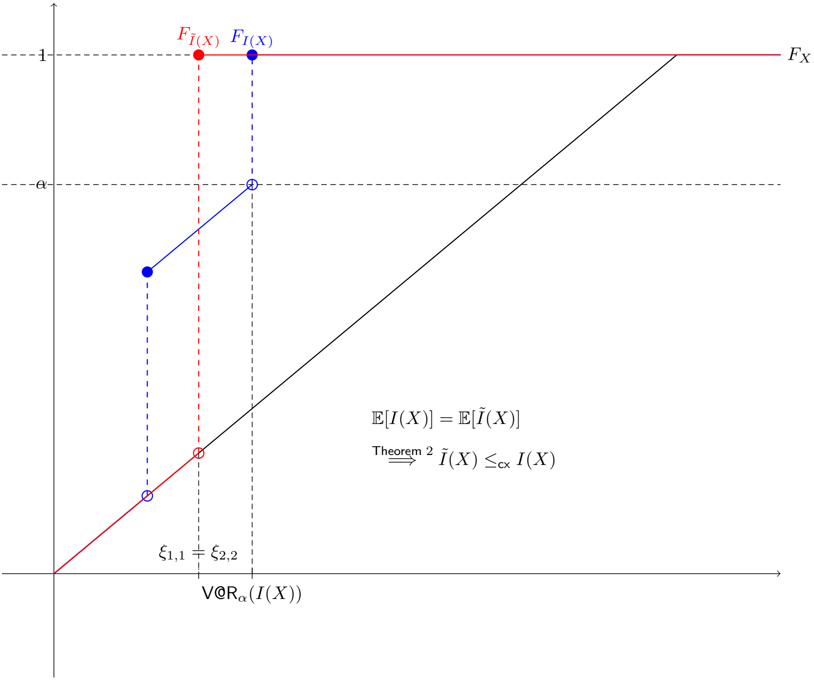
Assume that G solely depends on the first argument, such that $J_1(G) = 1$ but $J_2(G) = J_3(G) = 0$. The optimal ceded loss function for the problem takes the form of

$$I^*(x) = x - (x - d^*)_+,$$

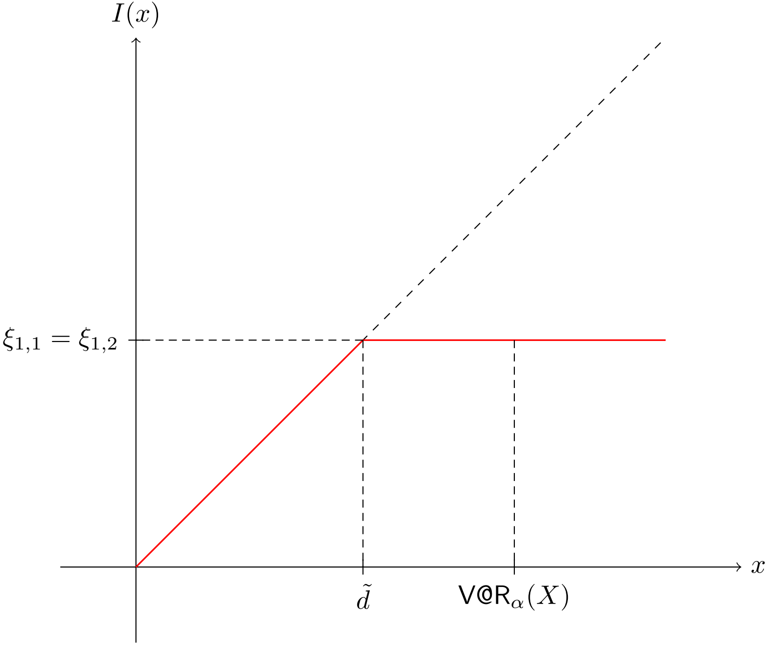
for some $0 \leq d^* \leq \operatorname{ess\,sup} X$.



$F_{I(X)}$
1
F_X
α
$\mathsf{V@R}_\alpha(I(X))$



$F_{\tilde{I}(X)}$
$F_{I(X)}$
1
F_X
α
$\mathbb{E}[I(X)] = \mathbb{E}[\tilde{I}(X)]$
$\overset{\text{Theorem 2}}{\Longrightarrow} \tilde{I}(X) \leq_{\mathsf{cx}} I(X)$
$\xi_{1,1} = \xi_{2,2}$
$\mathsf{V@R}_\alpha(I(X))$



$\tilde{I}(x)$
$\xi_{1,1} = \xi_{1,2}$
x
$\tilde{d}$
$\mathsf{V@R}_\alpha(X)$

Conclusion and Future Potential Works

- We studied a broad class of optimal insurance decision problems, in which the objective involves the expectation, V@R and TV@R of the total loss exposure of the policyholder, while the premium principle of the insurance indemnity is characterized by its expectation and a functional preserving convex order, which includes well-known premium principles, such as the Wang's principle.
- We applied the Karlin-Novikoff-Stoyan-Taylor multiple-crossing conditions for convex ordering to solve the most general problem.
- We showed that the optimal solution of these problems in general takes the double layer form.
- We further solved sub-problems by making use of the solution of the most general problem and the Karlin-Novikoff once-crossing conditions for convex ordering.
- Future potential works: general risk measure in objective, V@R and TV@R of the insurer in premium, functional preserving other stochastic ordering in premium, ...

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Thank you!