Kolmogorov's forward PIDE and forward transition rates in life insurance

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Agenda

- Introduction
  - The classic life insurance setup: reserve and cash flow
- The doubly stochastic setup
  - A generalisation of Kolmogorov's forward differential equation
- Forward transition rates
- Cash flows: stochastic and deterministic payments
- Generalisation and the semi-Markov model

Talk is based on [Buchardt, 2016]

Introduction

Consider a pension policy

- Paid for by a single premium
- certain *guaranteed* benefits (life annuity, disability coverage, ...)

Consider two questions:

Question 1: What is todays value of the guarantee? Referred to as the *market (consistent) value* or simply *reserve*

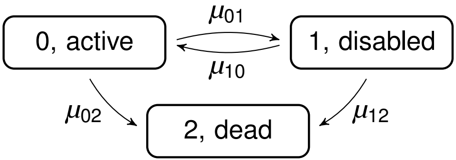
The answer depends on the market interest rate, hence:
Question 2: What is the sensitivity of the *market value* with respect to the interest rate? Easily calculated from the future expected *cash flow*

The classic setup: probabilities

Markov life insurance setup, [Norberg, 1991].

Payments occurs in states and upon transitions

- what is the probability of the payments occuring?



Probabilities calculated with *Kolmogorov's forward differential eq.*

$$\frac{\mathrm{d}}{\mathrm{d}s} p_{ij}(t,s) = -\sum_{\ell \neq j} p_{ij}(t,s)\mu_{j\ell}(s) + \sum_{\ell \neq j} p_{i\ell}(t,s)\mu_{\ell j}(s)$$

The classic setup: reserve and cash flow

Actual payments at time s

- if in state j: $b_j(s)$
- upon a transition $j \to \ell$: $b_{j\ell}(s)$

Expected payments at time s (the cash flow)

$$a_i(t,s) = \sum_{j \in \mathcal{J}} p_{ij}(t,s) \left(b_j(s) + \sum_{\ell \neq j} \mu_{j\ell}(s) b_{j\ell}(s) \right)$$

The reserve: present value of expected payments

$$V_i(t) = \int_t^\infty e^{-\int_t^s f(u)\,\mathrm{d}u} a_i(t,s)\,\mathrm{d}s$$

Remark: $V_i(t)$ also solves Thiele's ODE

Stochastic transition rates and payments

- Examples of stochastic transition rates
  - Stochastic mortality
  - Dependent mortality and disability
- Stochastic transition rates: A way to model dependence between policyholders
  - Stochastic mortality affects all policyholders
  - Mass surrender:
    - large customers/brokers moving to another pension provider
- Solvency II: Stochastic transition rates provide a way to model the risk
- Stochastic payments:
  - Benefits can be linked to the mortality level
  - Guaranteed annuity options

...and many other applications.

Doubly stochastic setup

Assume underlying stochastic process,

- $X(t)$ multi-dimensional diffusion process,

$$\mathrm{d}X(t) = \beta(t, X(t))\,\mathrm{d}t + \sigma(t, X(t))\,\mathrm{d}W(t)$$

- Stochastic transition rates: $\mu_{ij}(t, X(t))$
- Given the path of $X(t)$, assume $Z(t)$ is a Markov chain.

With stochastic transition rates:

- The reserve:
  - Thiele's ODE becomes a PDE, see [Dahl, 2004]
- The cash flow:
  - Kolmogorov's backward ODE becomes a PDE, see [Dahl, 2004]
  - Kolmogorov's forward ODE ...

In search of Kolmogorov's forward ...

Inspect Kolmogorov's forward differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} p_{ij}(t_0, t) \neq -\sum_{\ell \neq j} p_{ij}(t_0, t) \mu_{j\ell}(t, X(t)) + \sum_{\ell \neq j} p_{i\ell}(t_0, t) \mu_{\ell j}(t, X(t))$$

Doesn't directly generalise: the transition rates are stochastic.

Distribution of $X(t)$: the transition density

$$p(t_0, x_0, t, x)\,\mathrm{d}x = \Pr(X(t) = \mathrm{d}x \mid X(t_0) = x_0),$$

solves the Fokker-Planck PDE,

$$\frac{\partial}{\partial t} p(t_0, x_0, t, x) = -\sum_k \frac{\partial}{\partial x_k} \left(\beta_k(t, x) p(t_0, x_0, t, x)\right) + \frac{1}{2} \sum_{k,m} \frac{\partial^2}{\partial x_k \partial x_m} \left(\Sigma_{km}(t, x) p(t_0, x_0, t, x)\right)$$

where $\Sigma(t, x) = \sigma(t, x)\sigma(t, x)^\top$.

Kolmogorov's forward partial differential equation

Let $p_{ij}(t_0, x_0, t, x)$ be the joint conditional density of $Z(t), X(t)$,

$$p_{ij}(t_0, x_0, t, x)\,\mathrm{d}x = \Pr(Z(t) = j, X(t) = \mathrm{d}x | Z(t_0) = i, X(t_0) = x_0)$$

Theorem: Kolmogorov's forward PDE

$$\begin{aligned}
&\frac{\partial}{\partial t} p_{ij}(t_0, x_0, t, x) \\
&= -\sum_{\substack{\ell \in \mathcal{J} \\ \ell \neq j}} p_{ij}(t_0, x_0, t, x)\mu_{j\ell}(t, x) + \sum_{\substack{\ell \in \mathcal{J} \\ \ell \neq j}} p_{i\ell}(t_0, x_0, t, x)\mu_{\ell j}(t, x) \\
&\quad - \sum_{k=1}^{d} \frac{\partial}{\partial x_k} \left(\beta_k(t, x) p_{ij}(t_0, x_0, t, x)\right) \\
&\quad + \frac{1}{2} \sum_{k,m=1}^{d} \frac{\partial^2}{\partial x_k \partial x_m} \left(\Sigma_{km}(t, x) p_{ij}(t_0, x_0, t, x)\right)
\end{aligned}$$

Cash flows in the doubly stochastic setup

- Stochastic transition rates $\mu_{ij}(t, X(t))$
- Stochastic payments
  - In state i: $b_i(t, X(t))$
  - Jump from i to j: $b_{ij}(t, X(t))$

Theorem: The cash flow

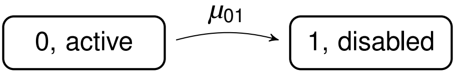
$$a_i(t_0, x_0, t) \\ = \sum_{j \in \mathcal{J}} \int p_{ij}(t_0, x_0, t, x) \Big(b_j(t, x) + \sum_{\ell \neq j} \mu_{j\ell}(t, x) b_{j\ell}(t, x) \Big) \, \mathrm{d}x$$

Compare with the classic setup

$$a_i(t_0, t) = \sum_{j \in \mathcal{J}} p_{ij}(t_0, t) \Big(b_j(t) + \sum_{\ell \neq j} \mu_{j\ell}(t) b_{j\ell}(t) \Big)$$

Forward transition rates: the mortality rate

- Consider the 2-state model



- Stochastic mortality rate $\mu_{01}(t, X(t))$

Forward mortality rate $g_{t_0}(t)$ (cf. [Milevsky and Promislow, 2001]),

$$p_{00}(t_0, t) = \mathrm{E}\left[e^{-\int_{t_0}^{t} \mu_{01}(s, X(s))\,\mathrm{d}s} \,\middle|\, X(t_0) \right] = e^{-\int_{t_0}^{t} g_{t_0}(s)\,\mathrm{d}s}$$

A “replacement result”.

Forward rates in the literature:

- Interest and mortality forw. rates, [Miltersen and Persson, 2005]
- *Dependent forward rates*, [Buchardt, 2014]
- General discussion, [Norberg, 2010]
- Consistent forw. rates: [Christiansen and Niemeyer, 2015]

Forward transition rates

Definition: Forward transition rate

The *forward transition rate* $f_{t_0}^{ij}(t)$, measured at time t_0, for the transition rate $\mu_{ij}(t, X(t))$ at time t, is defined as

$$f_{t_0}^{ij}(t) = \lim_{h \searrow 0} \frac{1}{h} \Pr\big(Z(t+h) = j \,|\, Z(t) = i, (Z, X)(t_0)\big)$$

Lemma: Forward transition rate as expectation

$$f_{t_0}^{ij}(t) = \mathrm{E}\left[\mu_{ij}(t, X(t)) \middle| Z(t) = i, (Z, X)(t_0)\right]$$

In the classic setup: Simplifies to the transition rates

$$\mu_{ij}(t) = \lim_{h \searrow 0} \Pr(Z(t+h) = j \mid Z(t) = i)$$

Forward transition rates

Proposition (replacement result – text version):

The forward transition rates are the transition rates of a Markov chain with the “same distribution” (as seen from time t_0).

The joint transition density of $(Z, X)(t)$:

$$p_{ij}(t_0, x_0, t, x)\,\mathrm{d}x = \Pr(Z(t) = j, X(t) = \mathrm{d}x | Z(t_0) = i, X(t_0) = x_0)$$

The transition density of $Z(t)$ only:

$$p_{ij}(t_0, x_0, t) = \int p_{ij}(t_0, x_0, t, x)\,\mathrm{d}x$$

Theorem: Kolmogorov’s forward ODE with forw. rates

$$\frac{\mathrm{d}}{\mathrm{d}t} p_{ij}(t_0, x_0, t) = \sum_{\ell;\ell\neq j} f^{\ell j}_{t_0,x_0}(t) p_{i\ell}(t_0, x_0, t) - \sum_{\ell;\ell\neq j} f^{j\ell}_{t_0,x_0}(t) p_{ij}(t_0, x_0, t)$$

Forward transition rate

Let $\tilde{Z}_{t_0}$ be a Markov chain with transition rates $f_{t_0}^{k\ell}(t)$.

Proposition (replacement result):

Conditional on $(Z, X)(t_0)$, we have $Z(t) \stackrel{D}{=} \tilde{Z}_{t_0}(t)$:

$$\begin{aligned}
&\Pr\left(\tilde{Z}_{t_0}(t) = k \mid X(t_0), Z(t_0) = \tilde{Z}_{t_0}(t_0) = \ell\right) \\
=\,&\Pr\left(Z(t) = k \mid X(t_0), Z(t_0) = \ell\right).
\end{aligned}$$

Can be used for expectations of $Z(t)$. Let g be some function,

$$\begin{aligned}
&\mathrm{E}\left[g(Z(t)) \mid (Z, X)(t_0) = (\ell, x_0)\right] \\
=\,&\mathrm{E}\left[g(\tilde{Z}_{t_0}(t)) \mid \tilde{Z}_{t_0}(t_0) = Z(t_0) = \ell, X(t_0) = x_0\right] \\
=\,&\sum_k p_{\ell k}(t_0, x_0, t) g(k).
\end{aligned}$$

Cash flows in the doubly stochastic setup

Stochastic payments $b_i(t, X(t))$, $b_{ij}(t, X(t))$

$$a_i(t_0, x_0, t) = \sum_{j \in \mathcal{J}} \int p_{ij}(t_0, x_0, t, x) \Big(b_j(t, x) + \sum_{\ell \neq j} \mu_{j\ell}(t, x) b_{j\ell}(t, x) \Big) \, \mathrm{d}x$$

Deterministic payments $b_i(t)$, $b_{ij}(t)$: payments only depend on $Z(t)$:

$$a_i(t_0, x_0, t) = \sum_{j \in \mathcal{J}} p_{ij}(t_0, x_0, t) \Big(b_j(t) + \sum_{\ell \neq j} f_{t_0}^{j\ell}(t) b_{j\ell}(t) \Big)$$

Essentially modelling with $\tilde{Z}_{t_0}(t)$.

Compare with the cash flow in the classic setup

$$a_i(t_0, t) = \sum_{j \in \mathcal{J}} p_{ij}(t_0, t) \Big(b_j(t) + \sum_{\ell \neq j} \mu_{j\ell}(t) b_{j\ell}(t) \Big)$$

Further generalisation and semi-Markov processes

Let $Y(t)$ be a multi-dimensional Markov jump-diffusion,

$$\mathrm{d}Y(t) = \beta(t, Y(t))\,\mathrm{d}t + \sigma(t, Y(t))\,\mathrm{d}W(t) + \mathrm{d}J(t).$$

(With jump intensity measure μ and $\rho(t,x) = \sigma(t,x)\sigma(t,x)^\top$.)

Let P be the transition probabilities,

$$\Pr(Y(t) \in A \mid Y(t') = x') = P(t, A; t', x') = \int_A P(t, \mathrm{d}x; t', x').$$

Theorem: Kolmogorov's forward PIDE

$$\begin{aligned}\frac{\partial}{\partial t} P(t, A; t', x') = &-\sum_i \int_A \frac{\partial}{\partial x_i}\left(\beta_i(t,x) P(t, \mathrm{d}x; t', x')\right) \\ &+ \frac{1}{2}\sum_{i,j} \int_A \frac{\partial^2}{\partial x_i \partial x_j}\left(\rho_{ij}(t,x) P(t, \mathrm{d}x; t', x')\right) \\ &+ \int_{A^\complement} \left(\int_A \mu(\mathrm{d}x; t, y)\right) P(t, \mathrm{d}y; t', x').\end{aligned}$$

Special cases of the general jump-diffusion

The doubly stochastic process: $(Z, X)(t) = Y(t)$

If $P(t, A; t', x')$ has a density $p(t, x; t', x')$, Kolmogorov's forward PDE exists for the density.

The semi-Markov process $(Z, U)(t) = Y(t)$

- U is the duration in the current state
- The transition probabilities of Z depends on U
- (No diffusion in this setup)

Here, $P(t, A; t', x')$ does not have a density with respect to the Lebesgue measure, since:

$$\Pr\left(U(t' + s) = u' + s \mid (Z, U)(t') = (k', u')\right) > 0.$$

However, the diffusion part does not exist, which leads to another simplification.

Kolmogorov's forward IDE for semi-Markov processes

Semi-Markov $(Z, U)(t)$ with transition probabilities

$$P(t,k,u;t',k',u') = \Pr\left(Z(t) = k, U(t) \leq u \mid Z(t') = k', U(t') = u'\right).$$

Theorem: Kolmogorov's forward integro-diff.-eq.

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial u}\right) P(t,k,u;t',k',u') = \sum_{\ell \neq k} \int_0^{u'+t-t'} \mu_{\ell k}(t,v) P(t,\ell,\mathrm{d}v;t',k',u') - \int_0^u \sum_{\ell \neq k} \mu_{k\ell}(t,v) P(t,k,\mathrm{d}v;t',k',u').$$

Special case of the Kolmogorov's forward PIDE. The result is well known from e.g. [Buchardt et al., 2015] or [Helwich, 2008].

Thank you!

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