Solvency Need Resulting from Provisioning Risk in an ORSA Context

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Own Risk and Solvency Assessment

Article 45 of the Solvency II Directive:

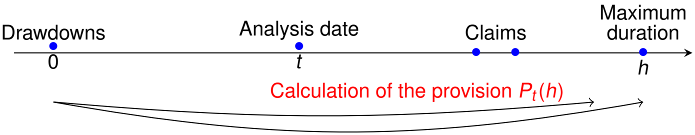
As part of its risk-management system every insurance undertaking [...] shall conduct its Own rRsk and Solvency Assessment [ORSA]. That assessment shall include at least [...] the overall solvency needs taking into account the specific risk profile, approved risk tolerance limits and the business strategy of the undertaking.

Solvency Need	Vs	Solvency Capital Requirement
1. Multi-year risk analysis,		1. One-year risk analysis,
2. Specific risks,		2. Risks imposed,
3. No risk measure imposed,		3. 99.5% Value At Risk,
4. Stress test required.		4. No stress test required.

==> Focus on the Solvency Need (SN) resulting from the provisioning risk (=reserving risk).

Provisioning risk

Context : insurance companies with long maturities contracts



where $P_t(h)$ is the provision evaluated at time t concerning contrats drawdown at date 0 and maturity less than h.

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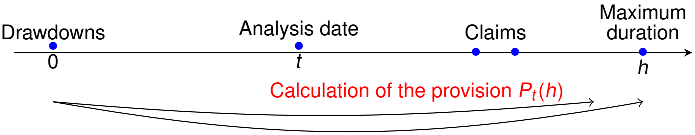

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Definition: provisioning risk

The RV measuring the provisioning risk is:

$$P_t(h) - \mathbb{E}[P_t(h)]. \tag{1}$$

==> it measures provisioning fluctuations.

See. "Uncertainty of the claims development result in the chain ladder method, Wuthrich, M.W. et al., 2009".

Provisioning modeling

We will denote:

- D_k (resp. T_k) the duration (resp. the maturity) of contracts of the underwriter k: if $T_k = D_k$ then there is no claim; so in the following: $0 < T_k < D_k < h$,
- M_k the capital at risk,
- $(R_x)_{x \geqslant 0}$ a stochastic process which represents a systemic risk identical for each underwriter.

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Definition

The provision evaluated at time t concerning contrats drawdown at date 0 and maturity less than h is:

$$\mathbf{P_t}(\mathbf{h}, \mathbf{R}) := \sum_{\mathbf{k}} \mathbf{g}\Big(\mathbf{M_k}\ \psi(\mathbf{T_k}, \mathbf{D_k}) - \mathbf{M_k} \mathbf{R}_{\mathbf{T_k}}\Big), \tag{2}$$

where:

- Ψ and g are explicit functions such as $g(x) = 0$ when $x < 0$,
- $M_k \psi(T_k, D_k)$ is the amount of the claim at date T_k, $M_k R_{T_k}$ is the amount of recoveries, and $g\Big(M_k\ \psi(T_k, D_k) - M_k R_{T_k}\Big)$ is the amount payed by the insurance company.

==> the model is based on individual behaviours of the underwriters and is different of the collectives models such as Chain Lader.

Solvency need

Solvency need

1. SN resulting from the provisioning risk: amounts that an insurance company needs to cover the risk of under-provisioning,
2. Risk measure: 99.5% Value At Risk for example.
3. Analysis period: maximum duration of contracts h.

SN at date t is defined by:

$$SN_t(h) := Q_{P_t(h,R)-\mathbb{E}[P_t(h,R)]}(99.5\%), \tag{3}$$

where we adopt here the convention that Q_X stands for the quantile of the r.v. X.

Calculation

Two possibilities to calculate $SN_t(h)$:

1. Empirical quantile (Law & Simulation),
2. Asymptotic quantile: (today)
   1. $P_t(h,R) - \mathbb{E}[P_t(h,R)] \xrightarrow[h \to +\infty]{\mathcal{L}} X$,
   2. law of X.

Main assumptions

Simplification: $M_k = 1$, $t = 0$ and $P_t(h, R) = P(h, R)$.

(A_1): The cloud of points $(T_k, D_k)_{k\geqslant 1}$ is a Poisson point process with intensity λLeb_{A_h}, where Leb_{A_h} is the Lebesgue measure restricted on $A_h = \{(t, d), 0 < t < d < h\}$.

(A_2): $(T_k, D_k)_{k\geqslant 1}$ and process $(R_x)_{x\geqslant 0}$ are independent,

(A_3): $(R_x)_{x\geqslant 0}$ is a stochastic process such as:

$$\lim_{x\to+\infty} R_x = 0, \text{a.s.}, \tag{4}$$

$$\int_0^{+\infty} R_x dx < +\infty, \text{a.s.}. \tag{5}$$

(A_4): The function $g : \mathbb{R} \to \mathbb{R}_+$ is continuous, K - Lipschitz, with $K \leqslant 1$, increasing on $\mathbb{R}_+$ and such as $g(x) = 0$ when $x < 0$.

Asymptotic quantile - General case

Theorem (Convergence in law)

With the previous assumptions and additional ones:

$$\frac{P(h,R) - \mathbb{E}[P(h,R)]}{h\rho_2} \xrightarrow[h \to +\infty]{\mathcal{L}} G - X^{\star} + \rho_1, \tag{6}$$

where:

1. ρ_1 *and* ρ_2 *are constants,*
2. *The RV* G *and* $X^{\star}$ *are independent,*
3. $G \sim N(0,1)$ *and:*

$$X^{\star} := \lambda \int_0^{+\infty} g^{\star}(x, R_x)dx, \tag{7}$$

where $g^{\star}$ *is an explicit function.*

==> So we approximate SN by:

$$SN^{\star}(h) := h\rho_2 Q_{G-X^{\star}+\rho_1}(99.5\%). \tag{8}$$

But it still remains theoretical!

Asymptotic quantile - Mortgage guarantee

In the case of mortgage guarantee $g(x) = \max(x, 0)$ and Ψ is explicit (link with exchange option), then:

$$X^\star = \lambda\rho_3 \int_0^{+\infty} \inf\{\rho_4, R_x\} dx, \tag{9}$$

where $\rho_3, \rho_4 > 0$: $X^\star$ is a perpetual integral functionals of Brownian motion.
We also assume:

$$R_x := \exp[\,\sigma B_x + \mu x], \tag{10}$$

with $\sigma > 0, \mu < 0$, and $(B_x)_{x \geqslant 0}$ a standard Brownian motion.

Simulation of *SN*

1. Remember Dufresne identity:

$$\int_0^{+\infty} R_x d_x \overset{(d)}{=} \frac{2}{\sigma^2}\frac{1}{\Gamma}, \tag{11}$$

   where Γ is a Gamma RV with parameters $-\frac{2\mu}{\sigma^2}$.

2. We can:
   1. combine Dufresne identity with the previous convergence results to simulate the law of $X^\star$,
   2. and then calculate SN.

Conclusion

1. General case:

$$\frac{P(h,R) - \mathbb{E}[P(h,R)]}{h\rho_2} \xrightarrow[h \to +\infty]{\mathcal{L}} G - X^\star + \rho_1. \quad (12)$$

where:

$$X^\star := \lambda \int_0^{+\infty} g^\star(x, R_x)dx. \quad (13)$$

2. Mortgage guarantee:

$$X^\star = \lambda\rho_3 \int_0^{+\infty} \inf\{\rho_4, R_x\}dx. \quad (14)$$

Dufresne identity:

$$\int_0^{+\infty} R_x d_x \overset{(d)}{=} \frac{2}{\sigma^2}\frac{1}{\Gamma}. \quad (15)$$

==> Evaluation of SN.
==> Stress test on the level of solvency.
==> Comparison with empirical approach.

3. To do:
    - Law of $X^\star$ in the general case (and for mortgage guarantee),
    - Other particular case: general liabilities, decennial guarantee.

References

- *“Individual loss provisioning with the multivariate skew normal distribution, Pigeon, M. et al., 2012”*
- *“Uncertainty of the claims development result in the chain ladder method, Wuthrich, M.W. et al., 2009”*
- *“An individual claims provisioning mode, Larsen, C.R. et al., 2007”*
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- *“Provisioning against borrowers default risk, P. Vallois et al., 2015”*
- *“ Asymptotic normality of shot noise on Poisson cluster processes with cluster marks”, Journal of Probability and Statistical, 2003, R. Serfling et al.*
- *“The distribution of a perpetuity, with application to risk theory and pension funding, Scandinavian Actuarial J., 1990”*

Annex

Empirical quantile

Provision law

The conditional law of $P(h, R)$ given that N_h is:

$$\Big[P(h,R)|N_h = n\Big] \overset{\mathcal{L}}{=} \Big[\sum_{k=1}^{n} g^{\star}\Big\{T_{(k)}, D_{(k)}, (G_i)_{1\leqslant i\leqslant k}\Big\}\Big], \tag{16}$$

where $g^{\star}$ is an explicit function, $(T_k, D_k)_{1\leqslant k\leqslant n}$ is uniformly distributed on the set A_h, $(G_k)_{1\leqslant k\leqslant n}$ are i.i.d $\mathcal{N}(0,1)$.

$\Rightarrow$ We can simulate the provision and obtain a l sample: $(p^1, ..., p^l)$.

Then the empirical Solvency Need, denoted by $SN^E(h)$, is:

$$SN^E(h) = \inf\{s : F_{P_t(h)}(s) \geqslant 99.5\%\} - \frac{1}{l}\sum_{i=1}^{l} p^i, \tag{17}$$

where $F_{P(h)}(s)$ is the empirical c.d.f. of the l-sample.

Empirical quantile - Proof - 1

Recall that N_s be the number of defaults on $]0, s]$ and:

$$N_s \overset{\mathcal{L}}{=} \mathcal{P}\Big[\lambda_2 \frac{(h-s)^2}{2}\Big].$$

1. We assume that $N_h - N_t = 2$ and we ordinate the date of default T_1 and T_2 as follows:

   $$T_{(0)} := 0 < T_{(1)} < T_{(2)}.$$

   The term and the amount associated to the loan which defaulted at date $T_{(k)}$ are denoted by $D_{(k)}$ and $M_{(k)}$ respectively. So:

   $$P_t(h,R) \overset{\mathcal{L}}{=} g\Big[M_{(1)}(\psi_1(T_{(1)}, D_{(1)}) - \psi_2(R_{T_{(1)}}))\Big] + g\Big[M_{(2)}(\psi_1(T_{(2)}, D_{(2)}) - \psi_2(R_{T_{(2)}}))\Big].$$

2. Let $0 < t_1 < t_2$. We have:

   $$\Big[R_{t_2} | R_{t_1} = x_1\Big] \overset{\mathcal{L}}{=} \Big[x_1 R_{t_2 - t_1}\Big].$$

   So:

   $$\Big(R_{T_{(1)}}, R_{T_{(2)}}\Big) \overset{\mathcal{L}}{=} \Big[\exp(\sigma\sqrt{T_{(1)}}G_1 + \mu T_{(1)}), \prod_{i=1}^{2} \exp(\sigma\sqrt{T_{(i)} - T_{(i)-1}}G_i + \mu(T_{(i)} - T_{(i)-1}))\Big].$$

   where G_1, G_2 are gaussian i.i.d..

3. In addition, random subscripts (1) and (2) do not depend on T_1 and T_2. Yet $(M_k)_{1\leqslant k\leqslant 2}$ are independent of $(T_k)_{1\leqslant k\leqslant 2}$, so:

   $$\Big[M_{(1)}, M_{(2)}\Big] \overset{\mathcal{L}}{=} \Big[M_1, M_2\Big].$$

Therefore:

$$\Big[P_t(h,R) | N_h - N_t = 2\Big] \overset{\mathcal{L}}{=} g\Big\{M_{(2)}(\psi_1(T_{(1)}, D_{(1)}) - \psi_2(\exp(\sigma\sqrt{T_{(1)}}G_1 + \mu(T_{(1)}))))\Big\}$$

$$+ g\Big\{M_2(\psi_1(T_{(2)}, D_{(2)}) - \psi_2(\prod_{i=1}^{2} \exp(\sigma\sqrt{T_{(i)} - T_{(i)-1}}G_i + \mu(T_{(i)} - T_{(i)-1}))))\Big\}.$$

Empirical quantile - Proof - 2

Proof:
Let $0 < t_1 < ... < t_n$. We have:

$$\Big[R_{t_n} | R_{t_1} = x_1, ..., R_{t_{n-1}} = x_{n-1}\Big] \overset{\mathcal{L}}{=} \Big[x_{n-1} R_{t_n - t_{n-1}}\Big]. \tag{18}$$

So:

$$\Big[\Big(R_{T_{k_1}}, ..., R_{T_{k_n}}\Big)\Big] \overset{\mathcal{L}}{=} \Big[\Big(\exp(\sigma\sqrt{T_{k_1}} G_1 + \mu T_{k_1}), ..., \prod_{l=1}^{n} \exp(\sigma\sqrt{T_{k_l} - T_{k_l - 1}} G_k + \mu(T_{k_l} - T_{k_l - 1}))\Big)\Big]. \tag{19}$$

Combining (18) and (19) it follows:

$$\Big[P_t(h,R) | N_t = n\Big] \overset{\mathcal{L}}{=} \Big[\sum_{k=1}^{n} g\Big\{M_{k_k}\Big[\psi_1(T_{k_k}, D_{k_k}) - \psi_2(\prod_{l=1}^{k} \exp(\sigma\sqrt{T_{k_l} - T_{k_l - 1}} G_k + \mu(T_{k_l} - T_{k_l - 1})))\Big]\Big\}\Big]. \tag{20}$$

In addition, random subscripts $k_1, ..., k_n$ do not depend on $T_1, ..., T_n$. Yet $(M_k)_{1 \leqslant k \leqslant n}$ are independent of $(T_k)_{1 \leqslant k \leqslant n}$, so:

$$\Big[(M_{k_1}, ..., M_{k_n})\Big] \overset{\mathcal{L}}{=} \Big[(M_1, ..., M_n)\Big]. \tag{21}$$

$$\Big[P_t(h,R) | N_t = n\Big] \overset{\mathcal{L}}{=} \Big[\sum_{k=1}^{n} g\Big\{\Big[M_k \psi_1(T_{k_k}, D_{k_k}) - \psi_2(\prod_{l=1}^{k} \exp(\sigma\sqrt{T_{k_l} - T_{k_l - 1}} G_k + \mu(T_{k_l} - T_{k_l - 1})))\Big]\Big\}\Big].$$

□

Asymptotic quantile (3)

Algorithm 0.1 Approximation of the solvency need $SN^\star(h)$.

1: {**Step 1**} Let $\epsilon > 0$, $\beta \in]0,1[$, and $\epsilon_1 := Q_\Gamma(\frac{1-\beta}{2})$, $\epsilon_2 := \frac{1}{2}\epsilon\epsilon_1\rho_4\sigma^2$, where Γ is a Gamma RV with parameters $-\frac{2\mu}{\sigma^2}$ and Q_Γ is its quantile function.
Recall that if $\mu < 0$, then there exists t_0, large enough such that:

$$\Phi\Big(\frac{\mu t_0 - \ln \epsilon_2}{\sqrt{t_0}}\Big) \leqslant \frac{1-\beta}{2}. \tag{22}$$

2: {**Step 2**} Let $n \geqslant 1$ and $l \geqslant 1$. Consider two independent families of RVs $(Z_{i,j}, 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant l)$ and $(G_j, 1 \leqslant j \leqslant l)$ such that:

- $(Z_{i,j}, 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant l)$ are iid and $Z_{i,j} \overset{(d)}{=} R_{\frac{t_0}{n}}$,
- $(G_j, 1 \leqslant j \leqslant l)$ are iid and $G_j \sim N(0,1)$.

3: {**Step 3**} Then $SN^\star(h)$ is approximated by $SN^{asymp}(h)$ where:

$$SN^{asymp}(h) := h\sqrt{C_2}\Big[\rho_3 + Q_{emp}(1-\beta)\Big], \tag{23}$$

where Q_{emp} stands for the quantile of the empirical distribution associated with:

$$\Big(G_k - \sqrt{2\lambda}\frac{t_0}{n}\sum_{i=1}^{n}\inf\Big\{1, \frac{1}{\rho_4}\prod_{j=1}^{i} Z_{j,k}\Big\}, 1 \leqslant k \leqslant l\Big). \tag{24}$$