Maximal dividends and ruin: Lagrange and beyond

Christian Hipp

Institute for Finance, Banking and Insurance (retired)
University of Karlsruhe

3rd EAJ Conference
September 6 - 8, 2016
Lyon, France

Contents

Contents

Basics for Lundberg models: ruin

Claim frequency λ, premium rate c, claim sizes $X, X_1, X_2, ...$

$$S(t) = s + ct - X_1 - ... - X_{N(t)},\ s \geq 0,$$

$$\tau = \inf\{t : S(t) < 0\}$$

$$\psi(s) = \mathbb{P}\{\tau < \infty | S(0) = s\}.$$

$$0 = \lambda E[\psi(s - X) - \psi(s)] + c\psi'(s),$$

dynamic equation.

Basics for Lundberg models: dividends

$$V(s) = \sup_D \left\{ E\left[\int_0^\infty e^{-\delta t} dD(t) | S(0) = s \right] \right\}$$

No dividend payments when $t \geq \tau$. Discount rate δ.

$$0 = -\delta V(s) + \lambda E[V(s-X) - V(s)] + cV'(s).$$

Unique solution $v(s)$ with $v(0) = v'(0) = 1$.

$$M = \arg\min v'(s).$$

$$V(s) = v(s)/v'(M),\ s \leq M,\ V(s) = V(M) + s - M,\ s \geq M.$$

Barrier strategy with barrier M.

The problem

$S^D(t) = S(t) - D(t)$ with dividend process;
$\tau^D, \psi^D(s)$ ruin time and probability.

$$V(s,\alpha) = \sup_D \left\{ E\left[\int_0^\infty e^{-\delta t} dD(t) | S(0) = s \right] : \psi^D(s) \le \alpha \right\} \quad (1)$$

with Lagrange multiplier $L > 0$:

$$V(s,L) = \sup_D \left\{ E\left[\int_0^\infty e^{-\delta t} dD(t) | S(0) = s \right] - L\psi^D(s) \right\} \quad (2)$$

The problem: Lagrange gap

$V(s,\alpha)$ is not equivalent to $V(s,L)$:

$$V(s,L) = \sup\{V(s,\alpha) - L\alpha, 0 \le \alpha \le 1\}.$$

There might be $\alpha_1 < \alpha_2$ satisfying

$$V(s,L) = V(s,\alpha_1) - L\alpha_1$$

$$V(s,L) = V(s,\alpha_2) - L\alpha_2.$$

For $\alpha_1 < \alpha < \alpha_2$ there does not exist $L(\alpha)$ with

$$V(s,L(\alpha)) = V(s,\alpha) - L(\alpha)\alpha.$$

The problem is hard

- Maximal dividends: ruin certain.
- Minimal ruin prob.: no dividends.
- Dividends discounted, ruin not.
- Stationary approach for (1), discrete model H (2003) computation; Lagrange gap; transfer to other models.
- Solution via non-stationary approach, Lagrange.
- Computation of corresponding ruin probabilities.

Contents

Definition

Consider ruin and dividends after time t, starting at $S(t) = s$. Dividends discounted to time 0.

$$E\left[\int_t^\infty e^{-\delta u} dD(u) | S^D(t) = s\right] \text{ dividends.}$$

$$\mathbb{P}\{S^D(u) < 0 \text{ for some } u \geq t | S^D(t) = s\} \text{ ruin.}$$

Useful for computations also in stationary models, and for problem (2) .

Dynamic equations

Lundberg generator:

$$\mathcal{G}f(s) = \lambda E[f(s-X) - f(s)] + cf'(s).$$

Ruin probability:

$$0 = V_t(s,t) + \mathcal{G}V(s,t). \tag{3}$$

Dividend value: in the no action region

$$s \le M(t) = \min\{s : V_s(s,t) \le e^{-\delta t}\}$$

use (3), and for $s \ge M(t)$

$$V(s,t) = V(M,t) + (s-M)e^{-\delta t}.$$

Classical ruin probability

Use (3) and choose T large, with initial conditions $V(s,t) = 1, s < 0,\ V(s,T) = 0, s \geq 0$:

$$V(s,T) - V(s,t) = \int_t^T V_t(s,u)du.$$

Discretisation with n steps from T to 0: $V(s,0)$ $n-$th partial sum of classical Beekman convolution formula. Gives good numerical approximation.
Similar results for dividend values.

Classical ruin probability: recursion

$$\begin{aligned}
\psi_n(s) &= \int_s^\infty \psi_n'(x)dx,\ s \geq 0,\\
\psi_n'(s) &= \frac{\lambda}{c}\left(\psi_{n-1}(s) - E[\psi_{n-1}(s-X)]\right),\ s \geq 0,\\
\psi_0(s) &= 1 \text{ for } s < 0,\ \psi_0(s) = 0 \text{ for } s \geq 0,\\
\psi_n(s) &= 1 \text{ for } s < 0.
\end{aligned}$$

Contents

Non-stationary setup

$$
\begin{aligned}
V(s,t) &= \sup_D \left\{ E\left[\int_t^\infty e^{-\delta u} dD(u) | S(0) = s \right] - L\psi^D(s,t) \right\}, \\
\psi^D(s,t) &= \mathbb{P}\{S^D(u) < 0 \text{ for some } u \ge t\}, \\
0 &= V_t(s,t) + \mathcal{G}V(s,t) \text{ in the no action region } s \le M(t), \\
V(s,t) &= V(M,t) + (s-M)e^{-\delta t}, \; s \ge M(t).
\end{aligned}
$$

Boundary values:

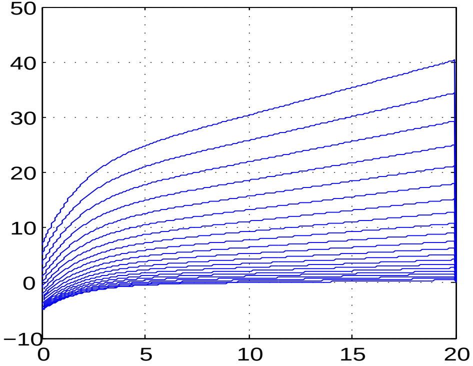
$$
\begin{aligned}
V(s,T) &= -L\psi(s), \\
V(s,t) &= -L, s < 0.
\end{aligned}
$$

Discretisation for Lundberg

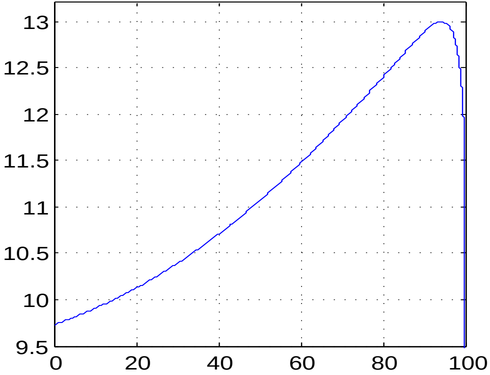
$$\begin{aligned}
V(s, t - dt) &= V(s, t) + dt\, \mathcal{G} V(s, t), \\
V(s, t - dt) &= V(s - ds, t - dt) + ds\, e^{-\delta t}, \\
E[V(s - X, t)] &= \sum_{i=1}^{k} V(s - i\, ds, t) \mathbb{P}\{(i-1)\, ds \leq X < i\, ds\} \\
V'(s, t) &= (V(s, t) - V(s - ds, t))/ds.
\end{aligned}$$

$$T = 100,\ dt = 0.001,\ ds = 0.01,\ 0 \leq s \leq 20.$$

$V(s, t)$ for Lundberg

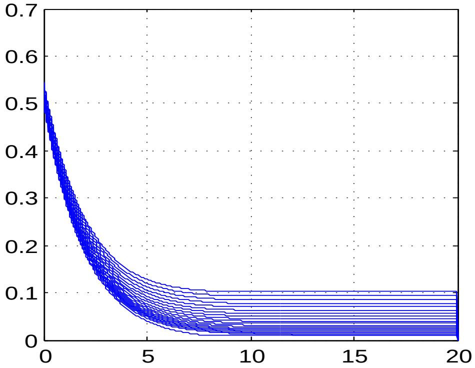


$M(t)$ for Lundberg



$M(t)$ decreasing for t close to T, typical.

$\psi(s, t)$ for Lundberg



Results for a Diffusion

$$dS(t) = dt + dW(t),\ \ t \geq 0.$$

Boundary condition:

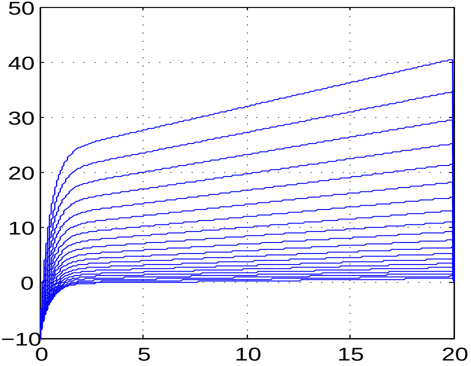
$$V(s,t) = 0,\ \ s \leq 0.$$

Discretization: $ds = 0.01,\ \ dt = (ds)^2 = 0.0001,$

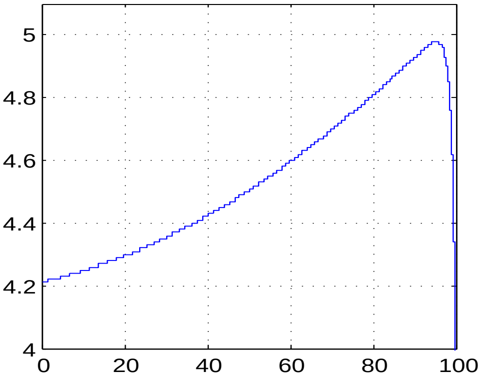
$$V'(s,t) = (V(s,t) - V(s-ds,t))/ds,$$

$$V''(s,t) = (V'(s+ds,t) - V'(s,t))/ds.$$

$V(s,t)$ for Diffusion

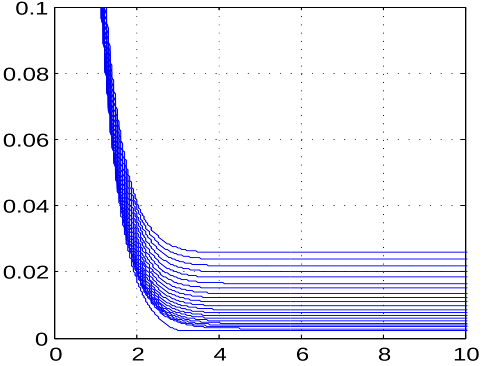


$M(t)$ for Diffusion



$M(t)$ decreasing for t close to T, typical.

$\psi(s, t)$ for Diffusion



Findings

- at the boundary of the no action region, dividends are paid at rate $c - M'(t) < c$
- barriers not constant between claims (H 2016b)
- improvement procedure in H (2016a) almost optimal
- dividend control under ruin constraint with reinsurance and investment control (discrete example)

Literature

1. Albrecher, Thonhauser 2007 IME
2. Dickson, Waters 2004 ASTIN Bull
3. Gerber, Shiu, Smith 2006 ASTIN Bull
4. Hernandez, Junca 2015 IME
5. Hernandez, Junca 2016
6. Hipp 2003 Blaetter DGVFM
7. Hipp 2016a Festschrift Ragnar Norberg
8. Hipp 2016b ASTIN Lisbon
9. Schmidli 2007 Springer