

Optimal Cash Management Control for Processes with two-sided Jumps

Nora Muler

Universidad Torcuato Di Tella (UTDT), Buenos Aires

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Coauthor: Pablo Azcue (UTDT)

Consider a continuous time cash management problem of a firm where the uncontrolled money stock X_t is a compound Poisson process with two-sided jumps and a negative drift m ;

$$X_t = x + mt + \overbrace{\sum_{i=1}^{N_t^1} (-U_i^1)}^{\text{Negative Jumps}} + \overbrace{\sum_{j=1}^{N_t^2} U_j^2}^{\text{Positive Jumps}}.$$

We assume:

- 1 The arrival Poisson processes N_t^1 corresponding to the negative jumps and N_t^2 corresponding to positive jumps are independent of each other with intensity β_1 and β_2 respectively.
- 2 The sizes of the negative and positive jumps are iid random variables with distributions F_1 and F_2 independent of each other as well.

A Brownian motion can also be added to the uncontrolled process. We consider in this presentation, for simplicity sake, the one without Brownian motion.

In this optimization problem the aim is to minimize the expected discounted cost: The money manager continuously monitors the cashflow and at any time he/she can increase or decrease the amount of cash.

The cost has two components: the opportunity cost of holding cash and the cost coming from decreasing and increasing the amount of cash.

We consider the case that both the cost of increasing and decreasing the amount of cash have proportional and fixed components; the fixed components lead to an impulse control problem.

The control strategy should be adapted and consists on both the times and amounts of downward and upward adjustments of cash: $\pi = (\pi^d, \pi^u)$,

- $\pi^d = (\bar{\xi}_n^d, Z_n^d)_{n \geq 0}$, $\bar{\xi}_n^d$ are the times and Z_n^d the amount of downward adjustments.
- $\pi^u = (\bar{\xi}_n^u, Z_n^u)_{n \geq 0}$ and $\bar{\xi}_n^u$ are the times and Z_n^u the amounts of upward adjustments .

The corresponding controlled process is:

$$X_t^\pi = X_t - \overbrace{\sum_{n=1}^{\infty} I_{\{\bar{\xi}_n^d \leq t\}} Z_n^d}^{\text{Downward Adjustments}} + \overbrace{\sum_{n=1}^{\infty} I_{\{\bar{\xi}_n^u \leq t\}} Z_n^u}^{\text{Upward Adjustments}}.$$

We assume that it is mandatory to inject cash in order that $X_{t^+}^\pi \geq 0$ for all t . Let us call $K_d^0 > 0$ and $K_d^1 \geq 0$ the fixed and proportional cost coefficient of downward adjustment and $K_u^0 > 0$ and $K_u^1 \geq 0$ the ones of upward adjustments.

Let $\delta > 0$ be the discount rate. The expected discounted cost has two terms.

1. The total expected discounted opportunity cost of holding cash:

$$H_{\pi}(x) = E_x\left(\int_0^{\infty} e^{-\delta t} aX_t^{\pi} dt\right), \text{ where } a > 0.$$

1. The total expected discounted opportunity cost $H_\pi(x)$ of holding cash:

$$H_\pi(x) = E_x\left(\int_0^\infty e^{-\delta t} aX_t^\pi\right)dt, \text{ where } a > 0.$$

2. The total expected discounted cost $A_\pi(x)$ coming from adjustments:

An upward adjustment Z_n^d of cash at time $\bar{\xi}_n^d \Rightarrow$ a cost $K_d^0 + K_d^1 Z_n^d$.

A downward adjustment Z_n^u of cash at time $\bar{\xi}_n^u \Rightarrow$ a cost $K_u^0 + K_u^1 Z_n^u$.

$$A_\pi(x) = E_x\left(\overbrace{\sum_{n=1}^{\infty} e^{-\delta \bar{\xi}_n^d} (K_d^0 + K_d^1 Z_n^d)}^{\text{Discounted cost of downward adjustments}} + \overbrace{\sum_{n=1}^{\infty} e^{-\delta \bar{\xi}_n^u} (K_u^0 + K_u^1 Z_n^u)}^{\text{Discounted cost of upward adjustments}}\right)$$

So the value function of an strategy π is

$$C_\pi(x) = H_\pi(x) + A_\pi(x).$$

And our aim is to find $C(x) = \inf_\pi C_\pi(x)$ and the optimal strategy π^* (if it exists).

Properties of $C(x)$:

- $C(x) \leq K_d^1 x + K_0$ (linear growth condition) and it is Lipschitz.
- It is a viscosity solution of the associated Hamilton-Jacobi-Bellman (HJB) equation:

$$\min \left\{ L(C)(x), \min_{0 \leq w \leq x} (K_d^1 w + K_d^0 + C(x - w)) - C(x), \min_{z \geq 0} (K_u^1 z + K_u^0 + C(z + x)) - C(x) \right\} = 0$$

where

$$L(C)(x) = mC'(x) - (\beta_1 + \beta_2 + \delta)C(x) + \beta_1 \int_0^\infty C(x - \alpha) dF_1(\alpha) + \beta_2 \int_0^\infty C(x + \alpha) dF_2(\alpha) + ax$$

We will see that c' could not exist at some points.

HJB equation and the optimal strategy.

If the current cash reserve x satisfies the equality

$$\min_{0 \leq w \leq x} (K_d^1 w + K_d^0 + C(x - w)) - C(x) = 0,$$

the optimal strategy would be a downward adjustment.

Calling

$$Z^d = \arg \min_{0 \leq w \leq x} (K_d^1 w + K_d^0 + C(x - w)).$$

the cash reserve after the adjustment becomes $x - Z^d$ paying $K_d^1 Z^d + K_d^0$ as a transaction cost.

We also have,

$$\int_{x-Z^d}^x (C'(s) - K_d^1) ds = K_d^0$$

If the current cash reserve x satisfies

$$\min_{z \geq 0} (K_u^1 z + K_u^0 + C(z + x)) - C(x) = 0,$$

the optimal strategy would be an upward adjustment.

Calling

$$Z^u = \arg \min_{0 \leq z} (K_u^1 z + K_u^0 + C(z + x)).$$

the cash reserve after the adjustment becomes $x + Z^u$ paying $K_u^1 Z^u + K_u^0$ as a transaction cost.

We also have,

$$\int_x^{x+Z^u} (C'(s) + K_u^1) = -K_u^0$$

Otherwise, if the current cash reserve x satisfies

$$L(C)(x) = 0,$$

where

Infinitesimal Generator of the Compound-Poisson Process with drift

$$L(C)(x) = \underbrace{mC'(x) - (\beta_1 + \beta_2)C(x) + \beta_1 \int_0^\infty C(x - \alpha) dF_1(\alpha) + \beta_2 \int_0^\infty C(x + \alpha) dF_2(\alpha)}_{\text{Infinitesimal Generator of the Compound-Poisson Process with drift}}$$

Discounted Term

$$\underbrace{-\delta C(x)}$$

+

Opportunity Cost Term

$$\underbrace{ax}$$

then nothing should be done (non-intervention region).

So: the HJB equation suggests the optimal strategy.

Important Results:

- $C(x)$ is the largest viscosity solution satisfying the linear growth condition (characterization result).
- If there is a cost function of an admissible strategy $C_\pi(x)$ that is a viscosity solution of the HJB equation, then π is the optimal strategy and $C(x) = C_\pi(x)$. (Verification Result).

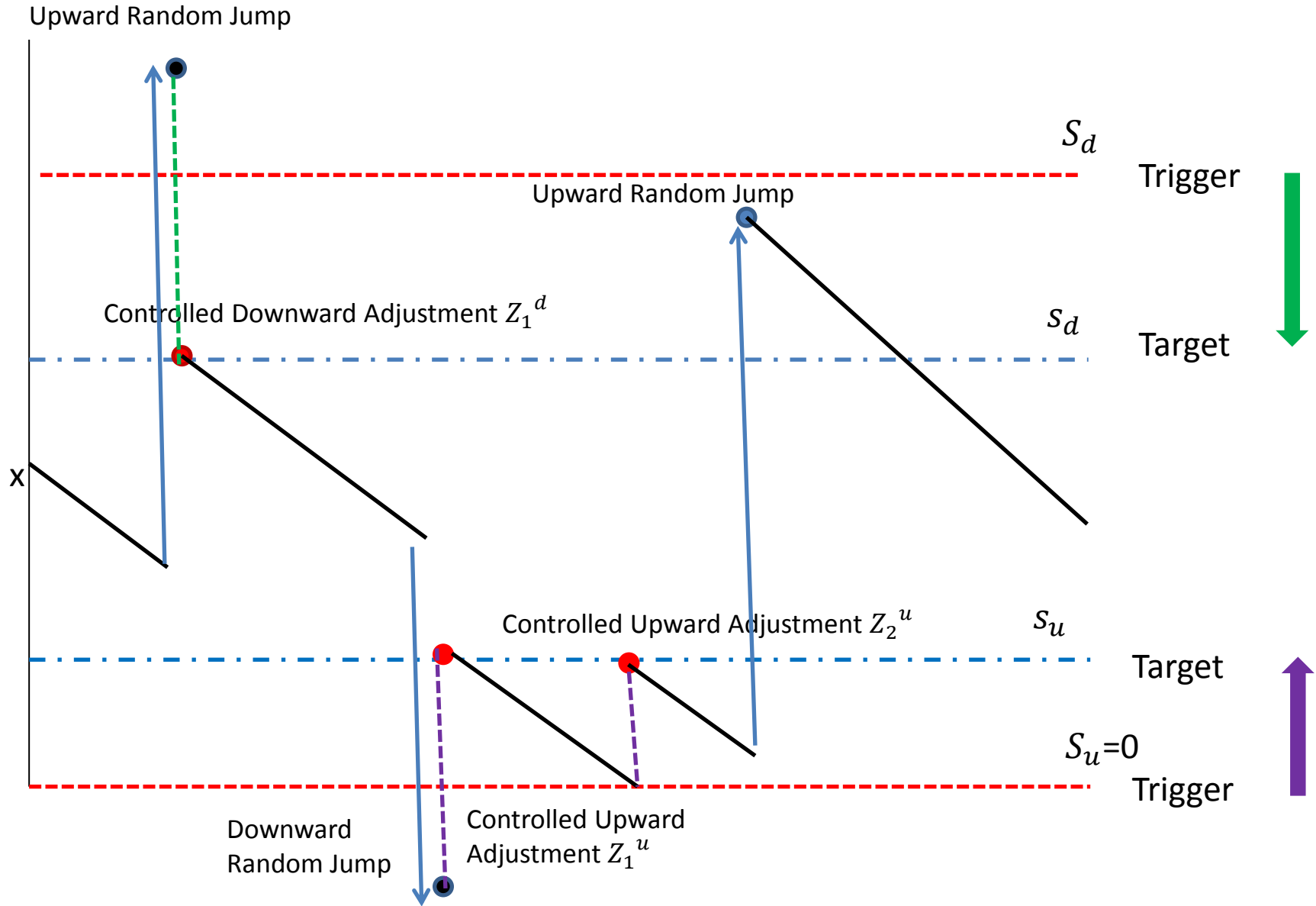
What we expect from the structure of the optimal strategy?

One of the simplest candidates for optimal strategies are the so-called impulse band strategies (IBS) that are characterized by:

- Two trigger levels S_d and S_u .
- Two targets levels s_d and s_u , where $S_d > s_d \geq s_u > S_u$.

Whenever the cash level is greater than S_d it is instantaneously decreased to s_d and whenever the cash level is less than S_u the cash amount is instantaneously increased to s_u .

Simulated Trajectory under an Impulse Band Strategy (S_d, s_d, s_u, S_u)



Results about the optimal strategy

- The optimal strategy exists and it depends only on the current cashflow.
- There is always a unique upward trigger $S_u = 0$ and target $S_u > 0$. S_u is always zero because there is a positive opportunity cost of holding cash and because the required minimum level is zero.
- If $K_d^1 \geq a/\delta$: there is no downward triggers; it is never optimal to make a downward adjustment, too expensive!.
- If $K_d^1 < a/\delta$ there are (finite) downward targets s_d^i and triggers S_d^i . The distance between consecutive downward targets and triggers is never less than $K_d^0 / (a/\delta - K_d^1)$.
- If $K_d^1 = 0$ there is a unique downward target s_d (that corresponds to the point that the minimum of $C(x)$ is attained). This implies a unique downward trigger? **NO!**, we will see an example.

We are thinking conditions under which the optimal strategy is an impulse band strategies (IBS).

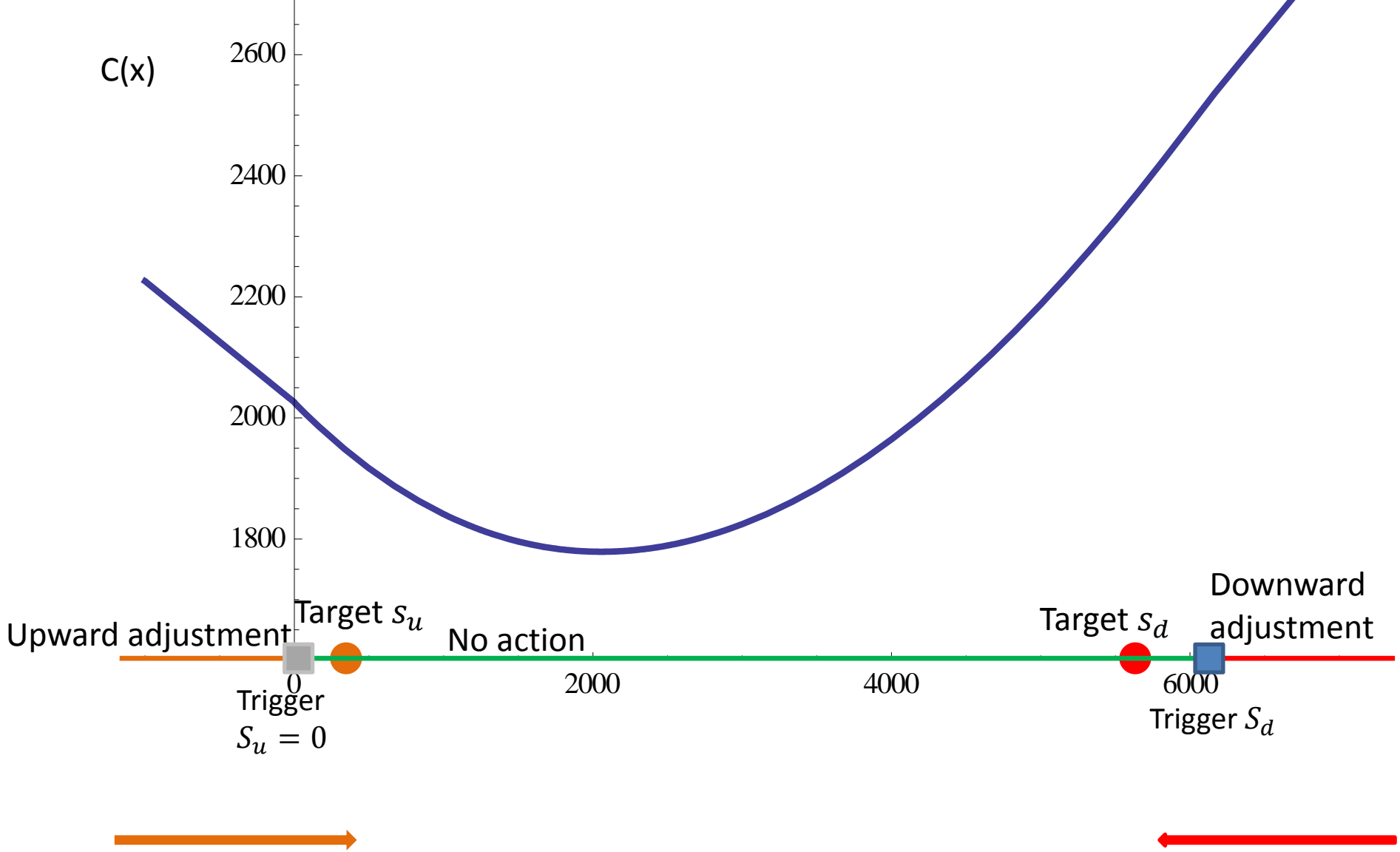
Numerical Example with an IBS optimal strategy

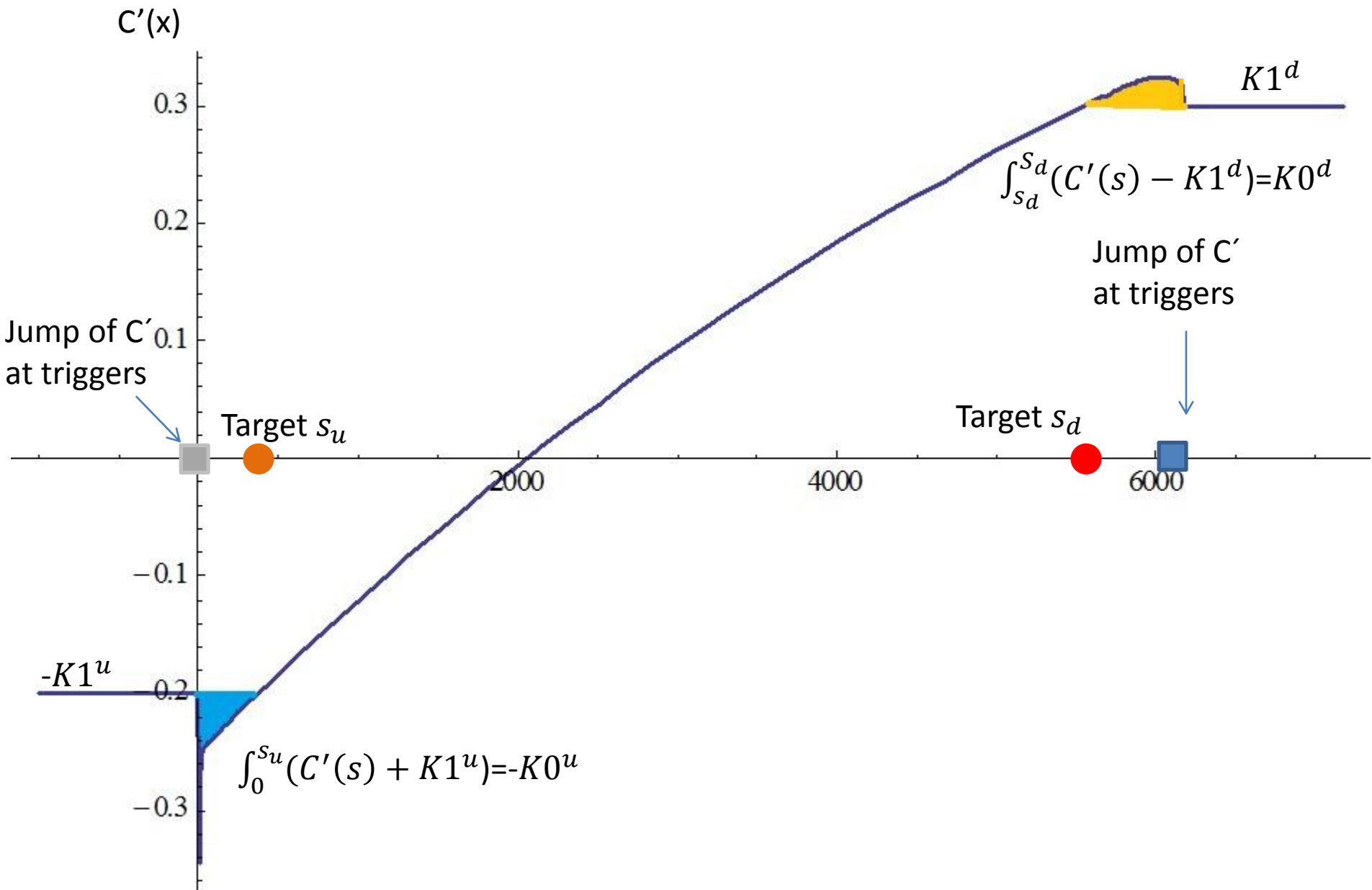
Parameters of the example:

- * Downward and upward jumps with exponential distribution functions:
 $F_1(x) = 1 - e^{-0.05x}$, $F_2(x) = 1 - e^{-0.02x}$.
- * Discount factor: $\delta = 0.06$.
- * drift: $m = -365$.
- * Poisson intensity: $\beta_1 = 30, \beta_2 = 10$.
- * $a = 0.05, K_u^1 = 0.2, K_d^1 = 0.3, K_u^0 = K_d^0 = 1$.

We have that:

- o The optimal strategy is an impulse band strategies (IBS) with
 $S_u = 0, s_u = 372.3, s_d = 5551.7, S_d = 6172.4$





Numerical Example with two downward triggers

Parameters of the example:

- * Downward jumps with distribution function $F_1(x) = I_{x \geq 1}$ and exponential distribution for the upward jumps $F_2(x) = 1 - e^{-x}$.
- * Discount factor: $\delta = 0.1$.
- * drift: $m = -1$.
- * Poisson intensity: $\beta_1 = 10, \beta_2 = 1/100$.
- * $a = 20, K_u^1 = K_d^1 = 0$ (no proportional cost for cash adjustment), $K_u^0 = K_d^0 = 1$.

In this case the Optimal Strategy is not IBS!

- It has a unique "upward" trigger $S_u = 0$, and a unique target $S_u = 0.2$ (this is always the case).
- It has a unique downward target $S_d = 0.2$ (that is the minimum of $C(x)$ because the proportional cost is zero) BUT two downward triggers $S_u^1 = 1.0$ and $S_u^2 = 1.3$

$C(x)$

143.2

143.0

142.8

142.6

142.4

Upward Adj.

No Action

Down.
Adj.

No
Action

Downward Adjustment

0.5

0.5

1.0

1.5

2.0

TARGET: $s_d = s_u$

$Trigg.S1_d$

$Trigg.S2_d$



