Kolmogorov’s forward PIDE and forward transition rates in life insurance

Kristian Buchardt
http://kristian.buchardt.net

&

Department of Mathematical Sciences, University of Copenhagen

3rd European Actuarial Journal Conference
September 8, 2016
Agenda

- Introduction
  - The classic life insurance setup: reserve and cash flow
- The doubly stochastic setup
  - A generalisation of Kolmogorov’s forward differential equation
- Forward transition rates
- Cash flows: stochastic and deterministic payments
- Generalisation and the semi-Markov model

Talk is based on [Buchardt, 2016]
Introduction

Consider a pension policy

- Paid for by a single premium
- Certain *guaranteed* benefits (life annuity, disability coverage, ...)

Consider two questions:

**Question 1:** What is today's value of the guarantee? Referred to as the *market (consistent) value* or simply *reserve*

The answer depends on the market interest rate, hence:

**Question 2:** What is the sensitivity of the *market value* with respect to the interest rate? Easily calculated from the future expected *cash flow*
The classic setup: probabilities

Markov life insurance setup, [Norberg, 1991].

Payments occur in states and upon transitions

- what is the probability of the payments occurring?

0, active $\xrightarrow{\mu_{01}}$ 1, disabled

0, active $\xrightarrow{\mu_{02}}$ 2, dead

1, disabled $\xrightarrow{\mu_{10}}$ 2, dead

1, disabled $\xrightarrow{\mu_{12}}$ 2, dead

Probabilities calculated with Kolmogorov’s forward differential eq.

$$\frac{d}{ds} p_{ij}(t,s) = - \sum_{\ell \neq j} p_{ij}(t,s) \mu_{j\ell}(s) + \sum_{\ell \neq j} p_{i\ell}(t,s) \mu_{\ell j}(s)$$
The classic setup: reserve and cash flow

**Actual payments** at time $s$
- if in state $j$: $b_j(s)$
- upon a transition $j \rightarrow \ell$: $b_{j\ell}(s)$

**Expected payments** at time $s$ (the cash flow)

$$a_i(t, s) = \sum_{j \in J} p_{ij}(t, s) \left( b_j(s) + \sum_{\ell \neq j} \mu_{j\ell}(s) b_{j\ell}(s) \right)$$

The reserve: present value of expected payments

$$V_i(t) = \int_t^\infty e^{-\int_t^s f(u) \, du} a_i(t, s) \, ds$$

Remark: $V_i(t)$ also solves Thiele’s ODE
Stochastic transition rates and payments

- Examples of stochastic transition rates
  - Stochastic mortality
  - Dependent mortality and disability
- Stochastic transition rates: A way to model dependence between policyholders
  - Stochastic mortality affects all policyholders
  - Mass surrender:
    - large customers/brokers moving to another pension provider
- Solvency II: Stochastic transition rates provide a way to model the risk
- Stochastic payments:
  - Benefits can be linked to the mortality level
  - Guaranteed annuity options

...and many other applications.
Doubly stochastic setup

Assume underlying stochastic process,

- $X(t)$ multi-dimensional diffusion process,

\[
dX(t) = \beta(t, X(t)) \, dt + \sigma(t, X(t)) \, dW(t)
\]

- Stochastic transition rates: $\mu_{ij}(t, X(t))$
- Given the path of $X(t)$, assume $Z(t)$ is a Markov chain.

With stochastic transition rates:

- The reserve:
  - Thiele’s ODE becomes a PDE, see [Dahl, 2004]
- The cash flow:
  - Kolmogorov’s backward ODE becomes a PDE, see [Dahl, 2004]
  - Kolmogorov’s forward ODE ...
In search of Kolmogorov’s forward ...

Inspect Kolmogorov’s forward differential equation

\[
\frac{d}{dt} p_{ij}(t_0, t) = - \sum_{\ell \neq j} p_{ij}(t_0, t) \mu_{j\ell}(t, X(t)) + \sum_{\ell \neq j} p_{i\ell}(t_0, t) \mu_{j\ell}(t, X(t))
\]

Doesn’t directly generalise: the transition rates are stochastic.

Distribution of \( X(t) \): the transition density

\[
p(t_0, x_0, t, x) \, dx = \Pr(X(t) = dx \mid X(t_0) = x_0),
\]

solves the Fokker-Planck PDE,

\[
\frac{\partial}{\partial t} p(t_0, x_0, t, x) = - \sum_k \frac{\partial}{\partial x_k} (\beta_k(t, x) p(t_0, x_0, t, x))
\]

\[
+ \frac{1}{2} \sum_{k,m} \frac{\partial^2}{\partial x_k \partial x_m} (\Sigma_{km}(t, x) p(t_0, x_0, t, x))
\]

where \( \Sigma(t, x) = \sigma(t, x)\sigma(t, x)^\top \).
Kolmogorov’s forward partial differential equation

Let \( p_{ij}(t_0, x_0, t, x) \) be the joint conditional density of \( Z(t), X(t) \),

\[
p_{ij}(t_0, x_0, t, x) \, dx = \Pr(Z(t) = j, X(t) = dx | Z(t_0) = i, X(t_0) = x_0)
\]

Theorem: Kolmogorov’s forward PDE

\[
\frac{\partial}{\partial t} p_{ij}(t_0, x_0, t, x) \\
= - \sum_{\ell \neq j, \ell \in J} p_{ij}(t_0, x_0, t, x) \mu_{j\ell}(t, x) + \sum_{\ell \neq j, \ell \in J} p_{i\ell}(t_0, x_0, t, x) \mu_{\ell j}(t, x) \\
- \sum_{k=1}^{d} \frac{\partial}{\partial x_k} (\beta_k(t, x) p_{ij}(t_0, x_0, t, x)) \\
+ \frac{1}{2} \sum_{k, m=1}^{d} \frac{\partial^2}{\partial x_k \partial x_m} (\Sigma_{km}(t, x) p_{ij}(t_0, x_0, t, x))
\]
Cash flows in the doubly stochastic setup

- Stochastic transition rates $\mu_{ij}(t, X(t))$
- Stochastic payments
  - In state $i$: $b_i(t, X(t))$
  - Jump from $i$ to $j$: $b_{ij}(t, X(t))$

**Theorem:** The cash flow

$$a_i(t_0, x_0, t) = \sum_{j \in J} \int p_{ij}(t_0, x_0, t, x) \left( b_j(t, x) + \sum_{\ell \neq j} \mu_{j\ell}(t, x) b_{j\ell}(t, x) \right) dx$$

Compare with the classic setup

$$a_i(t_0, t) = \sum_{j \in J} p_{ij}(t_0, t) \left( b_j(t) + \sum_{\ell \neq j} \mu_{j\ell}(t) b_{j\ell}(t) \right)$$
Forward transition rates: the mortality rate

- Consider the 2-state model
  
  \[
  \begin{array}{c|c}
    0, \text{active} & \mu_{01} \\
    \hline
    1, \text{disabled} & \\
  \end{array}
  \]

- Stochastic mortality rate \( \mu_{01}(t, X(t)) \)

Forward mortality rate \( g_{t_0}(t) \) (cf. [Milevsky and Promislow, 2001]),

\[
\rho_{00}(t_0, t) = \mathbb{E} \left[ e^{-\int_{t_0}^{t} \mu_{01}(s, X(s)) \, ds} \mid X(t_0) \right] = e^{-\int_{t_0}^{t} g_{t_0}(s) \, ds}
\]

A “replacement result”.

Forward rates in the literature:

- Interest and mortality forw. rates, [Miltersen and Persson, 2005]
- \textit{Dependent forward rates}, [Buchardt, 2014]
- General discussion, [Norberg, 2010]
- Consistent forw. rates: [Christiansen and Niemeyer, 2015]
Forward transition rates

Definition: Forward transition rate
The \textit{forward transition rate} $f_{t_0}^{ij}(t)$, measured at time $t_0$, for the transition rate $\mu_{ij}(t, X(t))$ at time $t$, is defined as

$$f_{t_0}^{ij}(t) = \lim_{h \downarrow 0} \frac{1}{h} \Pr(Z(t + h) = j | Z(t) = i, (Z, X)(t_0))$$

Lemma: Forward transition rate as expectation

$$f_{t_0}^{ij}(t) = E[\mu_{ij}(t, X(t)) | Z(t) = i, (Z, X)(t_0)]$$

In the classic setup: Simplifies to the transition rates

$$\mu_{ij}(t) = \lim_{h \downarrow 0} \Pr(Z(t + h) = j | Z(t) = i)$$
Forward transition rates

Proposition (replacement result – text version):
The forward transition rates are the transition rates of a Markov chain with the “same distribution” (as seen from time $t_0$).

The joint transition density of $(Z, X)(t)$:

$$p_{ij}(t_0, x_0, t, x) \, dx = \Pr(Z(t) = j, X(t) = dx | Z(t_0) = i, X(t_0) = x_0)$$

The transition density of $Z(t)$ only:

$$p_{ij}(t_0, x_0, t) = \int p_{ij}(t_0, x_0, t, x) \, dx$$

Theorem: Kolmogorov’s forward ODE with forw. rates

$$\frac{d}{dt} p_{ij}(t_0, x_0, t) = \sum_{\ell; \ell \neq j} f_{t_0, x_0}^{\ell j} (t) p_{i\ell}(t_0, x_0, t) - \sum_{\ell; \ell \neq j} f_{t_0, x_0}^{j \ell} (t) p_{ij}(t_0, x_0, t)$$
Forward transition rate

Let \( \tilde{Z}_{t_0} \) be a Markov chain with transition rates \( f_{t_0}^{k\ell} (t) \).

**Proposition (replacement result):**

Conditional on \((Z, X)(t_0)\), we have \( Z(t) \overset{D}{=} \tilde{Z}_{t_0} (t) \):

\[
\Pr (\tilde{Z}_{t_0} (t) = k \mid X(t_0), Z(t_0) = \tilde{Z}_{t_0} (t_0) = \ell) = \Pr (Z(t) = k \mid X(t_0), Z(t_0) = \ell).
\]

Can be used for expectations of \( Z(t) \). Let \( g \) be some function,

\[
\mathbb{E} [g(Z(t)) \mid (Z, X)(t_0) = (\ell, x_0)] = \mathbb{E} [g(\tilde{Z}_{t_0} (t)) \mid \tilde{Z}_{t_0} (t_0) = Z(t_0) = \ell, X(t_0) = x_0] = \sum_k p_{\ell k} (t_0, x_0, t) g(k).
\]
Cash flows in the doubly stochastic setup

Stochastic payments \( b_i(t, X(t)), b_{ij}(t, X(t)) \)

\[
a_i(t_0, x_0, t) = \sum_{j \in J} \int p_{ij}(t_0, x_0, t, x) \left( b_j(t, x) + \sum_{\ell \neq j} \mu_{j\ell}(t, x) b_{j\ell}(t, x) \right) dx
\]

Deterministic payments \( b_i(t), b_{ij}(t) \): payments only depend on \( Z(t) \):

\[
a_i(t_0, x_0, t) = \sum_{j \in J} p_{ij}(t_0, x_0, t) \left( b_j(t) + \sum_{\ell \neq j} f_{t_0}^{j\ell}(t) b_{j\ell}(t) \right)
\]

Essentially modelling with \( \tilde{Z}_{t_0}(t) \).

Compare with the cash flow in the classic setup

\[
a_i(t_0, t) = \sum_{j \in J} p_{ij}(t_0, t) \left( b_j(t) + \sum_{\ell \neq j} \mu_{j\ell}(t) b_{j\ell}(t) \right)
\]
Further generalisation and semi-Markov processes

Let $Y(t)$ be a multi-dimensional Markov jump-diffusion,

$$dY(t) = \beta(t, Y(t)) \, dt + \sigma(t, Y(t)) \, dW(t) + dJ(t).$$

(With jump intensity measure $\mu$ and $\rho(t, x) = \sigma(t, x)\sigma(t, x)^\top$.)

Let $P$ be the transition probabilities,

$$\Pr(Y(t) \in A \mid Y(t') = x') = P(t, A; t', x') = \int_A P(t, dx; t', x').$$

**Theorem: Kolmogorov's forward PIDE**

$$\frac{\partial}{\partial t} P(t, A; t', x') = - \sum_i \int_A \frac{\partial}{\partial x_i} \left( \beta_i(t, x) P(t, dx; t', x') \right)$$

$$+ \frac{1}{2} \sum_{i,j} \int_A \frac{\partial^2}{\partial x_i \partial x_j} \left( \rho_{ij}(t, x) P(t, dx; t', x') \right)$$

$$+ \int_{A^c} \left( \int_A \mu(dx; t, y) \right) P(t, dy; t', x').$$
Special cases of the general jump-diffusion

The doubly stochastic process: \((Z, X)(t) = Y(t)\)

If \(P(t, A; t', x')\) has a density \(p(t, x; t', x')\), Kolmogorov’s forward PDE exists for the density.

The semi-Markov process \((Z, U)(t) = Y(t)\)

- \(U\) is the duration in the current state
- The transition probabilities of \(Z\) depends on \(U\)
- (No diffusion in this setup)

Here, \(P(t, A; t', x')\) does not have a density with respect to the Lebesgue measure, since:

\[
\Pr(U(t' + s) = u' + s \mid (Z, U)(t') = (k', u')) > 0.
\]

However, the diffusion part does not exist, which leads to another simplification.
Kolmogorov’s forward IDE for semi-Markov processes

Semi-Markov \((Z, U)(t)\) with transition probabilities

\[
P(t, k, u; t', k', u') = \Pr(Z(t) = k, U(t) \leq u \mid Z(t') = k', U(t') = u').
\]

Theorem: Kolmogorov’s forward integro-diff.-eq.

\[
\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial u} \right) P(t, k, u; t', k', u') = \sum_{\ell \neq k} \int_0^{u'+t-t'} \mu_{\ell k}(t, v) P(t, \ell, dv; t', k', u')
\]

\[
- \int_0^u \sum_{\ell \neq k} \mu_{k\ell}(t, v) P(t, k, dv; t', k', u').
\]

Special case of the Kolmogorov’s forward PIDE. The result is well known from e.g. [Buchardt et al., 2015] or [Helwich, 2008].
Thank you!
Dependent interest and transition rates in life insurance.

Kolmogorov’s forward PIDE and forward transition rates in life insurance.
*Scandinavian Actuarial Journal*.

Cash flows and policyholder behaviour in the semi-markov life insurance setup.

On the forward rate concept in multi-state life insurance.
References II

Stochastic mortality in life insurance: market reserves and mortality-linked insurance contracts.

Durational effects and non-smooth semi-Markov models in life insurance.
Doctoral dissertation, University of Rostock.

Mortality Derivatives and the Option to Annuitise.
References III

Is Mortality Dead? Stochastic Forward Force of Mortality Rate Determined by No Arbitrage.
(Working Paper), Norwegian School of Economics and Business Administration.

Reserves in life and pension insurance.

Forward mortality and other vital rates - Are they the way forward?