Capital allocation for portfolios with non-linear risk aggregation

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Contents

The Capital Allocation Problem

- Portfolio of I loss variables (lines of business)

$$\mathbf{X} = (X_1, \ldots, X_I)$$

- Aggregate loss

$$\sum_{i=1}^{I} X_i$$

- Aggregate capital determined using a risk measure

$$\rho\left(\sum_{i=1}^{I} X_i\right)$$

Risk measure properties

Assume throughout that ρ satisfies:

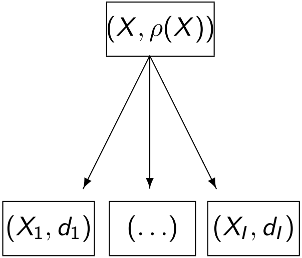
- Law invariance: $V_1 \stackrel{d}{=} V_2 \implies \rho(V_1) = \rho(V_2)$.
- Positive homogeneity: $\rho(aV_1) = a\rho(V_1)$ for $a \geq 0$.
- Subadditivity: $\rho(V_1 + V_2) \leq \rho(V_1) + \rho(V_2)$.
- Consistency with convex order: $V_1 \preceq_{cx} V_2 \implies \rho(V_1) \leq \rho(V_2)$.

Examples:

- Coherent distortion risk measures (Wang et al, 1997; Acerbi, 2002)
- Standard deviation

The Capital Allocation Problem

- How to distribute the benefits of diversification?
- Determine $d_1, \dots, d_I$ satisfying: $\sum_{i=1}^{I} d_i = \rho\left(\sum_{i=1}^{I} X_i\right)$



Literature streams

- Cooperative game theory (Denault, 2001; Tsanakas and Barnett, 2003; Kalkbrener, 2005; Csóka et al., 2009; Hougaard and Smilgins, 2016)
- Performance and portfolio management (Tasche, 1999; Buch et al., 2011)
- Market valuation (Myers and Read, 2001; Sherris, 2006; Zanjani, 2010; Bauer and Zanjani, 2015)
- Optimization (Dhaene et al., 2003; Dhaene et al., 2012)

The Gap

- For vector of exposures $\boldsymbol{w} = (w_1, \ldots, w_I)'$, loss from i^{th} line of business is $X_i(w_i)$. Capital as function of exposure:

$$r(\boldsymbol{w}) := \rho\left(\sum_{i}^{I} X_i(w_i)\right)$$

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- Standard assumption: portfolios are linear combinations of random variables, with the decision maker able to choose weights.

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- Mildenhall (2004, 2006): this assumption is not appropriate in an insurance context.

$$X_i(w_i) \neq w_i X_i(1), \quad r(\boldsymbol{w}) \neq \rho\left(\sum_i^I w_i X_i(1)\right)$$

A more flexible setup

- For $W > 1$, consider $\mathcal{F}_w$-adapted independent increasing Lévy processes
$$Y_i = \big(Y_i(w)\big)_{0 \le w \le W}, \quad i = 1, \dots, I$$
- For a $\mathcal{F}_W$-measurable random vector $(Z_1, \dots, Z_I)'$, independent of $(Y_1, \dots, Y_I)'$ define
$$X_i(w_i) = Y_i(w_i) + w_i Z_i, \qquad i = 1, \dots, I$$

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- If $Y_i(w_i) \equiv 0$, then $X_i(w_i) = w_i X_i(1)$.
  - Current capital allocation literature

- If $Z_i \equiv 0$, then X_i is a Lévy process
  - Actuarial risk models – if X_i is a Poisson process with unit intensity, then $X_i(w_i) \sim \mathrm{Poisson}(w_i)$.

The core of a fuzzy game

- Consider *fuzzy game* $r : [0, W]^I \mapsto \mathbb{R}$
- The *core* of r, $\mathcal{C}(r)$, is the set of vectors $\boldsymbol{d} \in \mathbb{R}^I$ such that

$$\sum_{i=1}^{I} w_i d_i \leq r(\boldsymbol{w}), \quad \forall \boldsymbol{w} \in [0,1]^I$$

$$\sum_{i=1}^{I} d_i = r(\mathbf{1}_I) = \rho\left(\sum_{i=1}^{I} X_i(1)\right)$$

- Core allocations give internal capital contribution rates that do not produce incentives for fragmentation

Euler allocations

- If r is positively homogeneous and partially differentiable at $\mathbf{1}_I$, the *Euler allocation* is:

$$\boldsymbol{d}^E(r) \in \mathbb{R}^I, \quad d_i^E(r) = \left.\frac{\partial r(\boldsymbol{w})}{\partial w_i}\right|_{\boldsymbol{w}=\mathbf{1}_I}.$$

- **Lemma** (Aubin, 1979): Let $r : [0, W]^I \mapsto \mathbb{R}$ be positively homogeneous and subadditive. Then, its core is non-empty. If r is partially differentiable at $\boldsymbol{w} = \mathbf{1}_I$, then the core is single-valued and $\boldsymbol{d}^E(r)$ is its unique element.

Two games

- The 'game actually played':

$$r(\boldsymbol{w}) = \rho\left(\sum_{i=1}^{I} X_i(w_i)\right)$$

  - Not generally homogeneous, can't apply Euler

- An auxiliary game:

$$\tilde{r}(\boldsymbol{w}) := \rho\left(\sum_{i=1}^{I} w_i X_i(1)\right)$$

  - Homogeneous, can apply Euler, but the 'wrong game'

Finding a core allocation for r

- **Lemma**: r is subadditive, $\tilde{r}$ is homogeneous and subadditive.
- **Proposition**: For $\boldsymbol{w} \in [0,1]^I$ it is $\tilde{r}(\boldsymbol{w}) \leq r(\boldsymbol{w})$.
- **Proposition**: $\mathcal{C}(\tilde{r}) \subseteq \mathcal{C}(r)$.
- **Corollary**: if $\boldsymbol{d}^E(\tilde{r})$ exists, then $\boldsymbol{d}^E(\tilde{r}) \in \mathcal{C}(r)$.

Example

- $I = 2$, X_1, X_2 are Poisson processes with unit intensity
- $X_1(w_1) + X_2(w_2) \sim \mathrm{Poisson}(w_1 + w_2)$.

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- $r(\boldsymbol{w}) = \sigma\big(X_1(w_1) + X_2(w_2)\big) = \sqrt{w_1 + w_2}$
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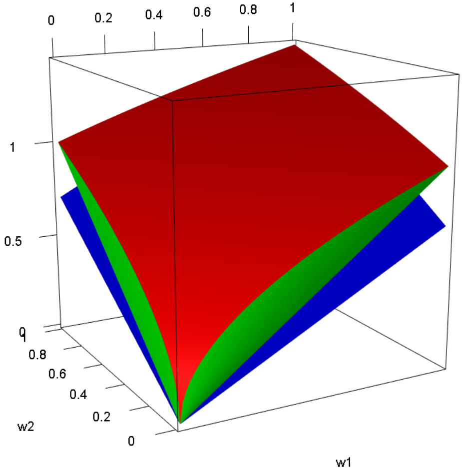
- Surfaces:

$$\boldsymbol{w} \mapsto (\boldsymbol{w}, r(\boldsymbol{w}))$$
$$\boldsymbol{w} \mapsto (\boldsymbol{w}, \tilde{r}(\boldsymbol{w}))$$
$$\boldsymbol{w} \mapsto \Big(\boldsymbol{w}, \boldsymbol{w}'\boldsymbol{d}^E(\tilde{r})\Big)$$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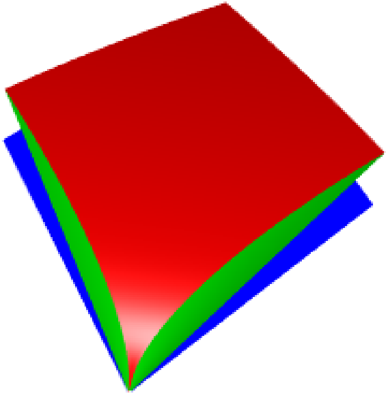
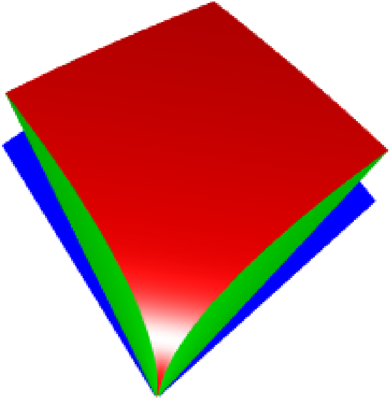
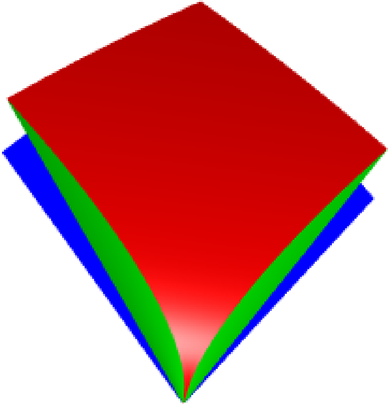
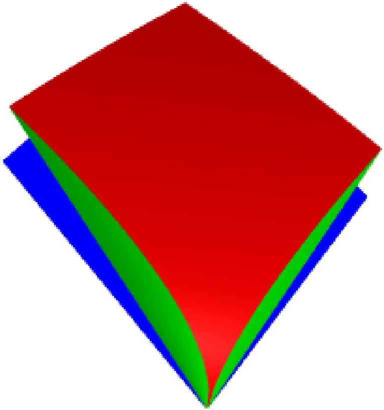
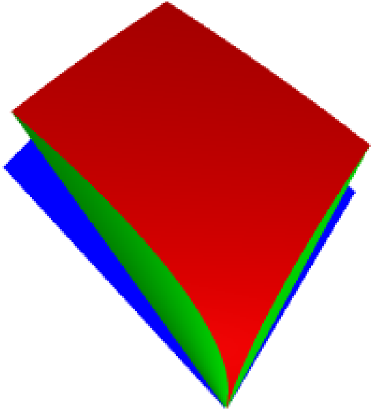
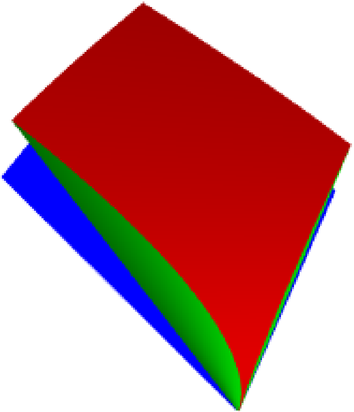


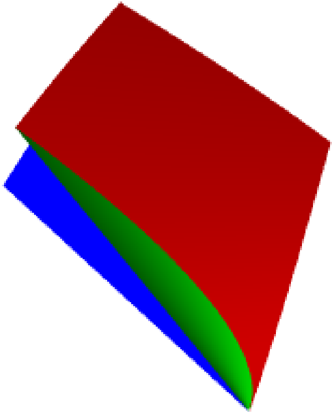


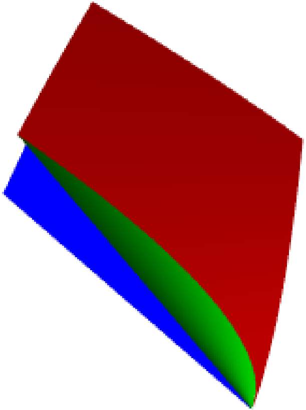


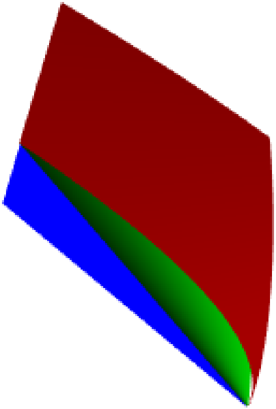


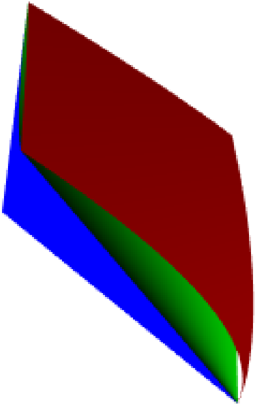


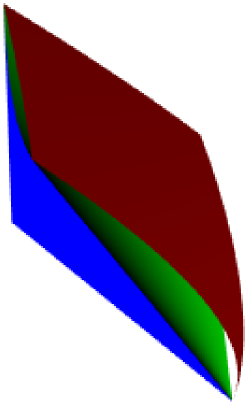


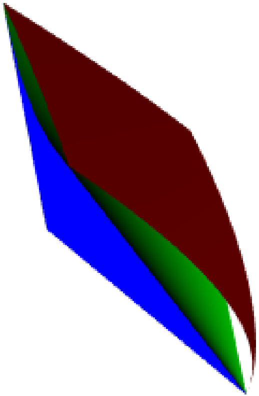


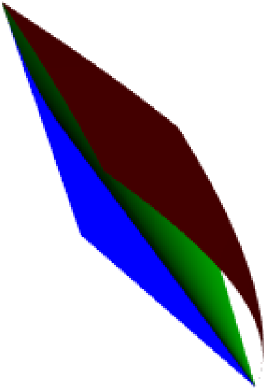


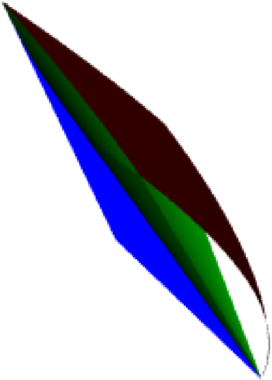


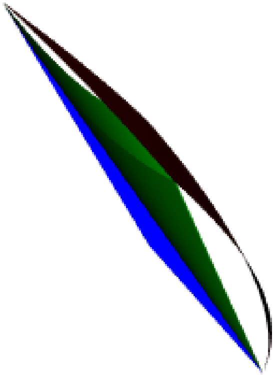


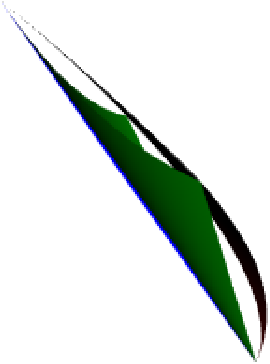






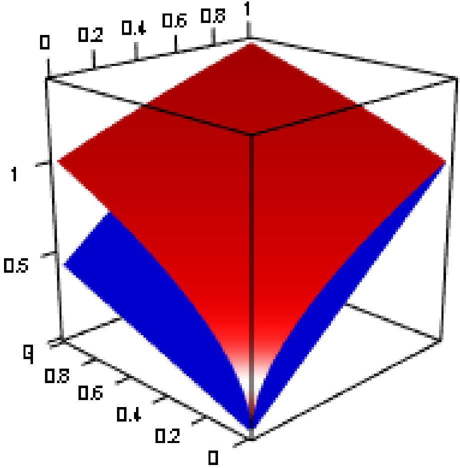






Uniqueness of core allocations

- $\boldsymbol{d}^E(\tilde{r})$, if it exists, is the only allocation $\mathcal{C}(\tilde{r})$
- Recall that $\mathcal{C}(\tilde{r}) \subseteq \mathcal{C}(r)$
- $\mathcal{C}(r)$ can contain more allocations
  - Precise characterisation available for concave r
  - Sufficient to consider corner points $\boldsymbol{w} \in \{0, 1\}^I$



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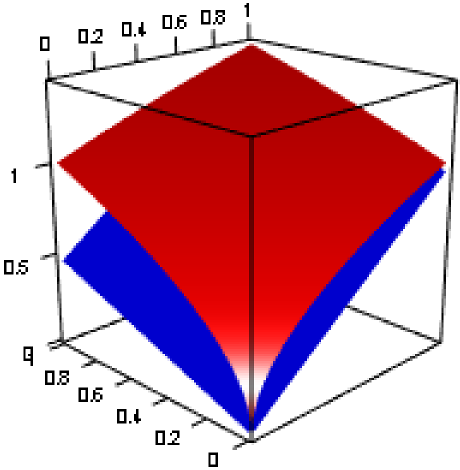
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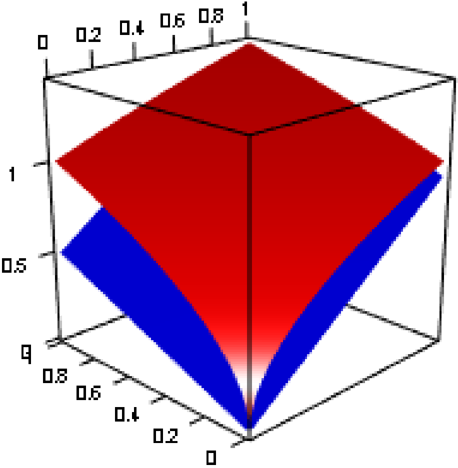
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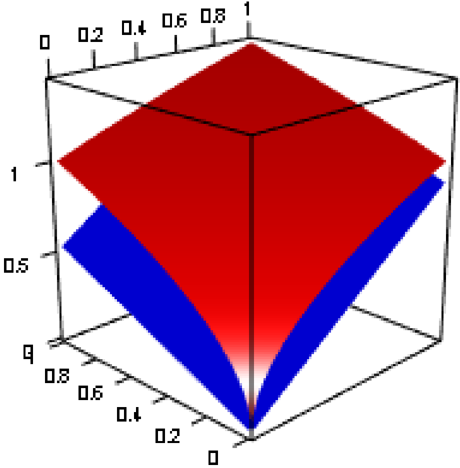

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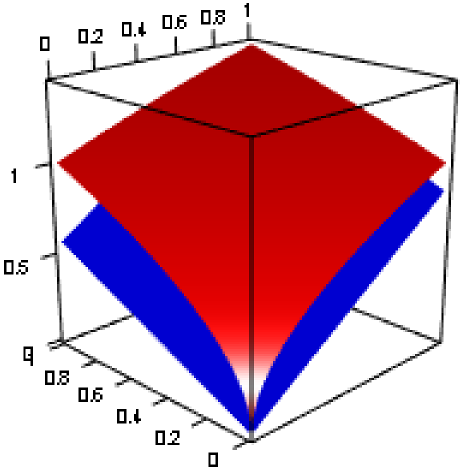
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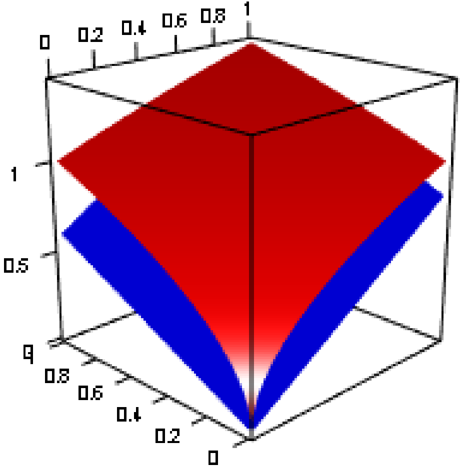
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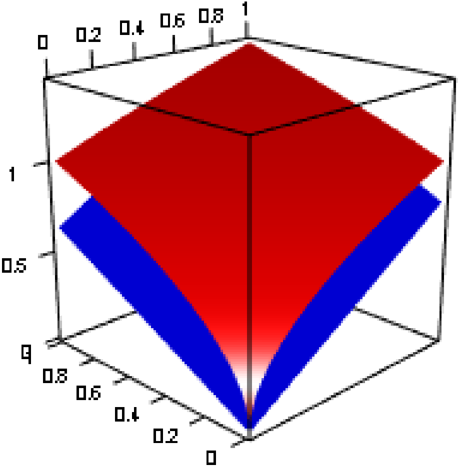

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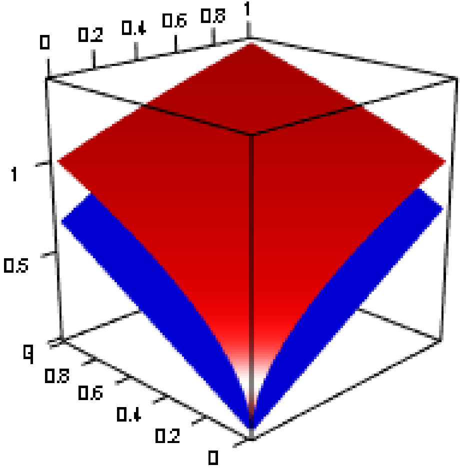
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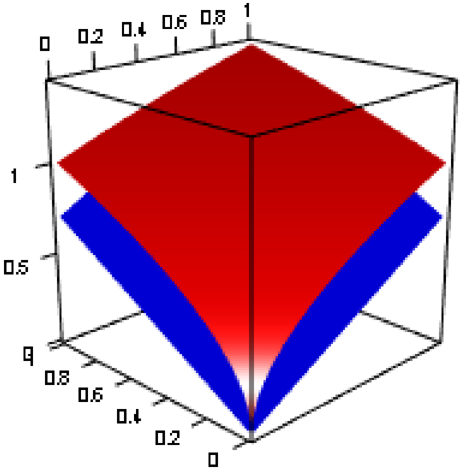

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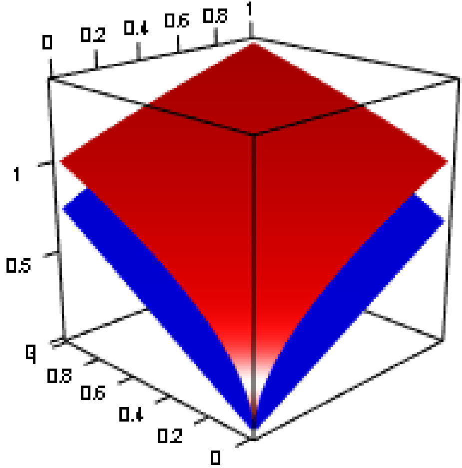
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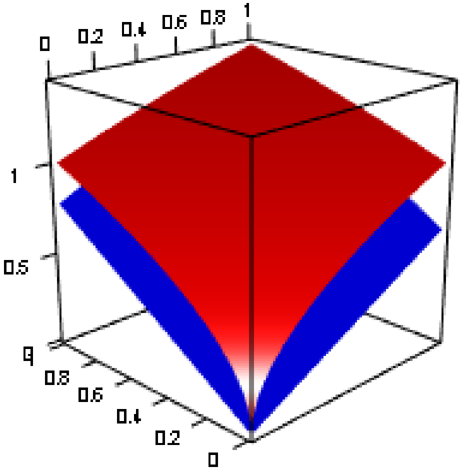

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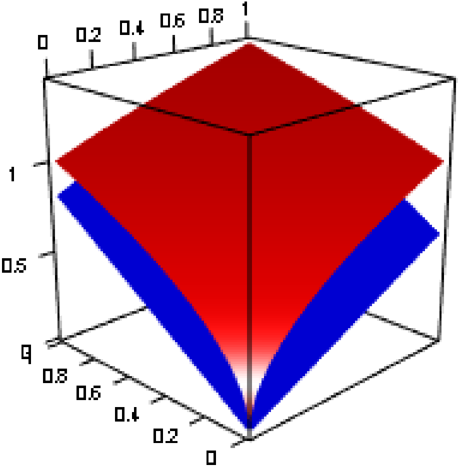
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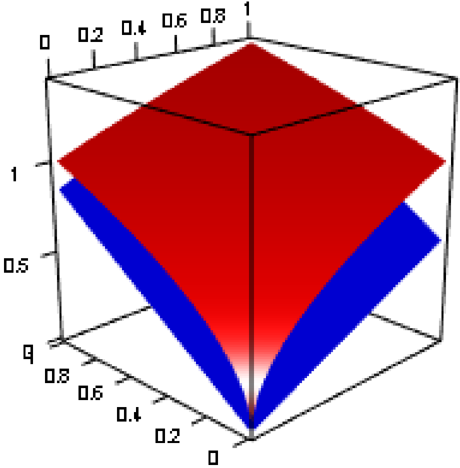

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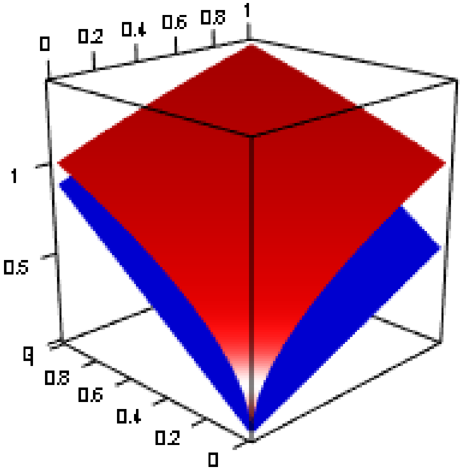
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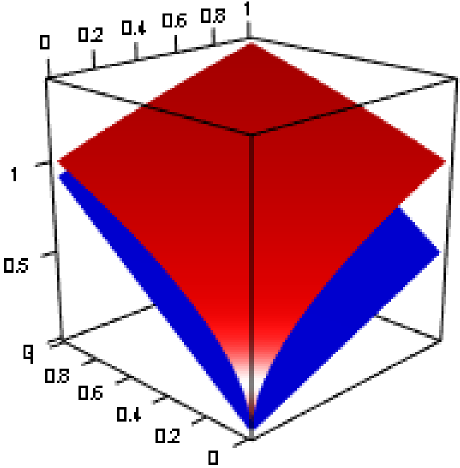
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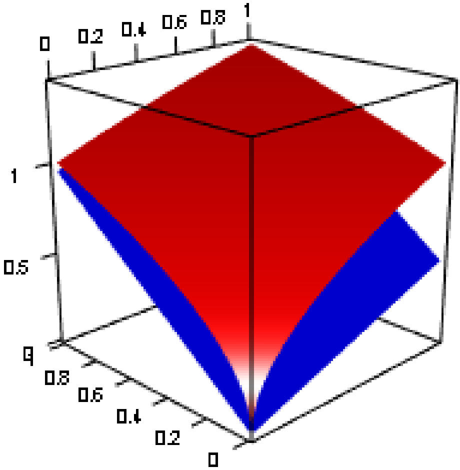
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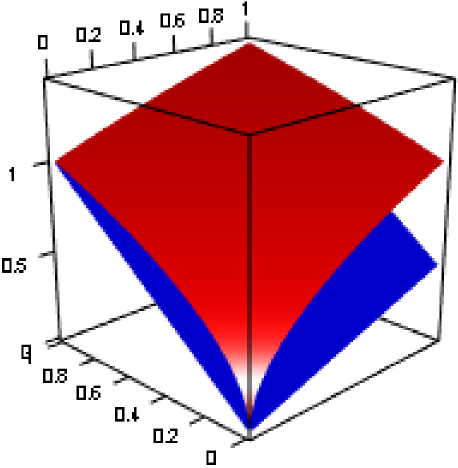
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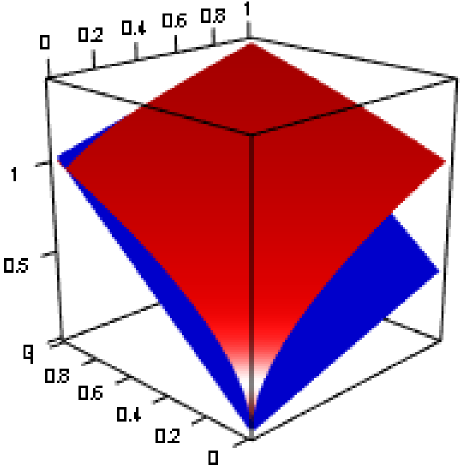
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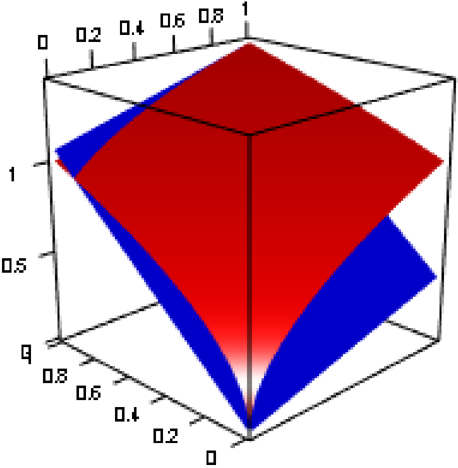
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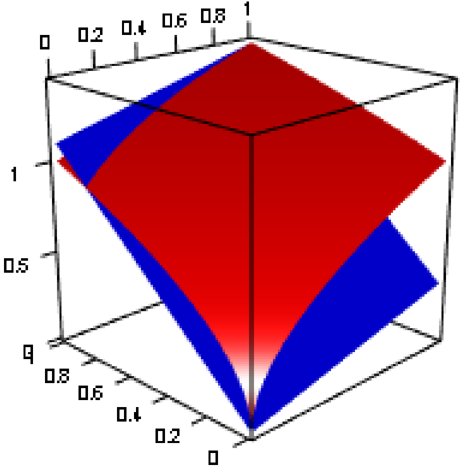

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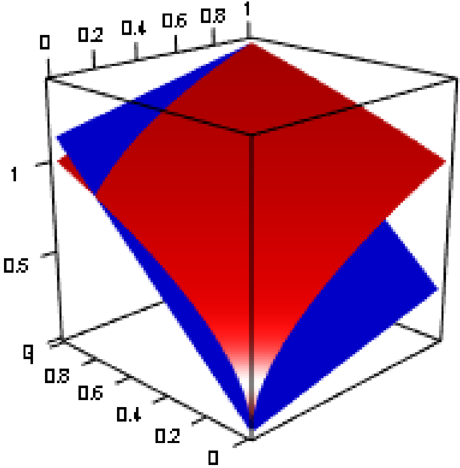
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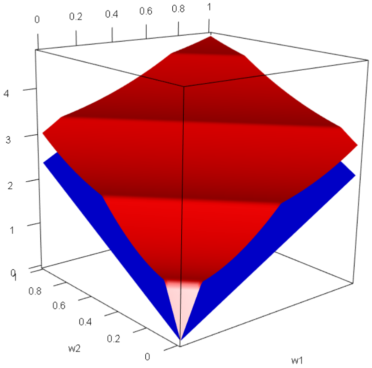
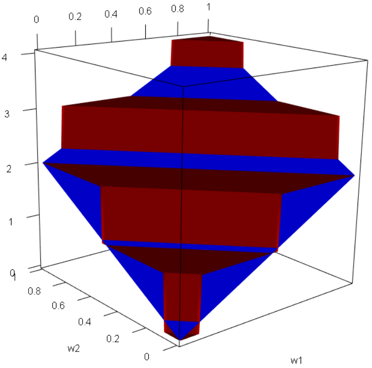
Suppose ρ is TVaR or VaR, each evaluated with $p = 0.9$.





Aumann-Shapley value

- The Aumann-Shapley allocation for the game r is given by

$$\boldsymbol{d}^{AS}(r) \in \mathbb{R}^I, \quad d_i^{AS}(r) = \int_0^1 \frac{\partial}{\partial w_i} r(\beta \mathbf{1}_I) d\beta$$

- Suggested for non-homogeneous risk capital allocation problems by Powers (2007).

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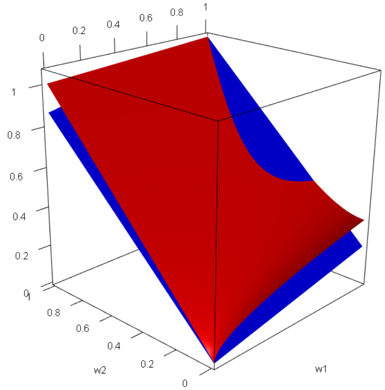
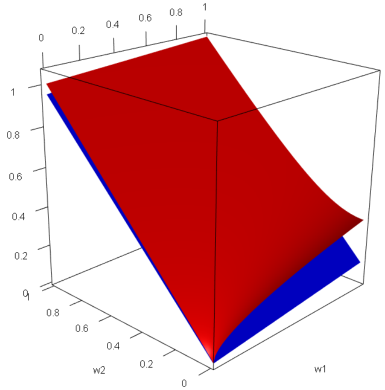
- **Theorem**: if r is concave and continuously differentiable, the Aumann-Shapley allocation $\boldsymbol{d}^{AS}(r)$ is in the fuzzy core.

Another example (left: Aumann-Shapley, right: $\boldsymbol{d}^E(\tilde{r})$):





Conclusion

- We study risk capital allocation problems for portfolios where risks are not aggregated linearly.
- In practice, the Euler method is often applied on the 'wrong' game $\tilde{r}$.
- We show that this is still a core-element of the 'right' game r.
- The Aumann-Shapley value can produce allocations outside the core.