A family of premium principles based on mixtures of TVaRs

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Outline

Premium Principles

- A premium principle is a rule that assigns an adequate price, called premium, to a risk to be insured.
- Formally, given a random variable representing a loss or an insurance risk X with distribution function F, a premium principle T assigns to X a number $T(X)$ which is the premium to be charged for accepting the risk X.
- A classical reference on this topic is the textbook by Goovaerts, De Vylder and Haezendonk (1984).

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Approaches to determine a premium principle (see Young, 2004)

One approach is to define $T(X)$ as simple as possible, by using certain quantities such as the expectation, the variance or other characteristics of X. Thus, some premium principles take the form

$$T(X) = E[X] + aD(X), \text{ for some } a \geq 0.$$

where $D(X)$ is a particular dispersion measure.

EXAMPLES:

- Standard deviation premium principle: $D(X) = \sqrt{\text{Var}[X]}$,
- Gini's premium principle: $D(X) = \frac{E|X_1 - X_2|}{2}$, where X_1 and X_2 are independent and have the same distribution as X),
- Denneberg's premium principle: $D(X) = E|X - Me|$, where Me denotes the median of X),
- Dutch premium principle: $D(X) = E\left[(X - \alpha EX)_+\right]$, with $\alpha \geq 1$.

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Approaches to determine a premium principle (see Young, 2004)

Another approach is to use a functional T satisfying a set of reasonable axioms:

- Independence: $T(X)$ depends only on the tail of X.
- Risk loading: $T(X) \geq E[X]$.
- No unjustified risk loading: If a risk X is identically equal to a constant $c \geq 0$ almost everywhere, then $T(X) = c$.
- No rip-off: $T(X) \leq \max[X] = F_X^{-1}(1)$.
- Translation invariance: $T(X + c) = T(X) + c$, for all risk X and constant c.
- Scale invariance: $T(cX) = cT(X)$, for all $c > 0$.
- Subadditivity: $T(X + Y) \leq T(X) + T(Y)$ for all risks X and Y.
- Comonotonic additivity: $T(X + Y) = T(X) + T(Y)$ for comonotonic X and Y.
- Stop-loss order preservation: If $X \leq_{sl} Y$, then $T(X) \leq T(Y)$.

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Approaches to determine a premium principle (see Young, 2004)

Definition

Given two risks X and Y with distribution functions F and G, respectively, X is said to be smaller than Y in the stop-loss order (denoted by $X \leq_{sl} Y$) if

$$TVaR_p(X) \leq TVaR_p(Y) \text{ for all } p \in (0,1),$$

where

$$TVaR_p(X) = \frac{1}{1-p} \int_p^1 F^{-1}(t)\,dt,\ \ p \in (0,1),$$

is the tail value-at-risk at level p of X.

Approaches to determine a premium principle (see Young, 2004)

A third approach is to derive the premium principle from a particular economic theory of risk.

EXAMPLE:

- A distortion premium principle is based on the idea of modifying the tail distribution $\overline{F}(x)$ with a distortion function h to get a risk-adjusted tail distribution $h\left(\overline{F}(x)\right)$ before taking the expectation.
- A distortion function h is a non-decreasing function $h : [0, 1] \longrightarrow [0, 1]$ with $h(0) = 0$ and $h(1) = 1$.
- When h is concave, the premium principle given by

$$T(X) = \int_0^{\infty} h\left(\overline{F}(x)\right) dx.$$

satisfies the properties (a) to (j).

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A new approach

In this work, given a risk distribution X, we use a different idea to obtain a risk-adjusted distribution Y and derive a premium principle.

STEPS:

1. We postulate some properties that the conditional distribution of our candidate risk-adjusted distribution Y, given the level of risk X, should satisfy.
2. We consider a distribution Y that satisfies these properties and define the premium principle $T_1(X) = E[Y]$.
3. This premium can be geometrically interpreted as the area under the curve defined by the tail value-at-risk $\mathbf{TVaR}_p(X)$:

$$T_1(X) = \int_0^1 \mathbf{TVaR}_p(X)dp.$$

4. We suggest a sequence of premium principles $T_{i,n}(X)$, where $n \geq 1$ and $1 \leq i \leq n$, that gradually incorporate the degree of risk aversion of the insurer in the safety loading.

A new approach

A risk-adjusted distribution Y is well-behaved under conditioning with respect to X if:

- **(P1)** The conditional distribution $[Y|X=x]$ only depends on the tail of X.

  Otherwise, the premium principle would not satisfy the independence property

- **(P2)** $E[Y|X=x] \geq x$ for all x.

  This ensures that Y satisfies the risk loading property since

$$E[Y] = \int_0^\infty E[Y|X=x] f(x)dx \geq \int_0^\infty xf(x)dx = E[X]$$

- **(P3)** $[Y|X=x_1] \leq_{st} [Y|X=x_2]$ for $x_1 < x_2$.

  This implies that $E[Y|X=x]$ is an increasing function of x (higher premiums for larger risks)

- **(P4)** If X is heavy tailed, the function $E[Y|X=x]-x$ increases as $x \to \infty$.

  This property protects, in the heavy tailed claim case, the insurer against very large claims.

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A new premium principle

Theorem

Let X be a risk with tail function $\overline{F}$. The random variable Y_X defined by the condition

$$[Y_X \,|X = x] \stackrel{d}{=} [X\,|X > x] \;\; \text{for all } x \geq 0$$

is a risk-adjusted distribution satisfying the properties P1, P2 and P3. Moreover, if $\overline{F}$ is log-convex, then Y_X also satisfies P4.

- $\overline{F}$ is log-convex if and only if X is DFR (decreasing failure rate), which is one way to formalize the notion “heavy tail”.
- Examples of DFR distributions are Gamma and Weibull distributions with shape parameters less than one, Pareto distribution and the convolution of exponential distributions.

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The Cumulative Residual Entropy Premium Principle

We study the premium principle

$$T_1(X) = E[Y_X] \quad \text{where} \quad [Y_X | X = x] \stackrel{d}{=} [X | X > x] \text{ for all } x \geq 0$$

Theorem

Let X be a random variable with tail function $\overline{F}$. Then,

$$T_1(X) = E[X] + \epsilon(X)$$

where

$$\epsilon(X) = -\int_0^{\infty} \overline{F}(x) \log \overline{F}(x) dx.$$

- The risk-loading $\epsilon(X)$ is called the cumulative residual entropy (Rao et al., 2004).

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PROPERTIES OF $\epsilon(X)$

- $\epsilon(X)$ is upper-bounded (Rao et al., 2004): $\epsilon(X) \leq \frac{E\left[X^2\right]}{2E[X]}$
- $\epsilon(X)$ is a variability measure in the sense of Bickel and Lehmann (1976), that is:

  (i) $\epsilon(X+k) = \epsilon(X)$ for all X and all constant k,
  (ii) $\epsilon(0) = 0$ and $\epsilon(\lambda X) = \lambda\epsilon(X)$ for all X and all $\lambda > 0$
  (iii) $\epsilon(X) \geq 0$ for all X, with $\epsilon(X) = 0$ if X is degenerated at c.
  (iv) $X \leq_{disp} Y$ implies $\epsilon(X) \leq \epsilon(Y)$.

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The CRE premium principle for some distributions

Example

- If X is exponential with mean λ, then $\epsilon(X) = \lambda$ and $T_1(X) = 2\lambda$.
- If X is uniform over $(0, a)$, then $\epsilon(X) = a/4$ and $T_1(X) = 3a/4$.
- If X is a Pareto distribution with parameters (α, β), then $\epsilon(X) = \alpha\beta/\left(\alpha - 1\right)^2$ and $T_1(X) = \beta/\left(\alpha - 1\right) + \alpha\beta/\left(\alpha - 1\right)^2$.

Three different representations

Remarks

- $T_1(X)$ takes the form
$$T_1(X) = E[X] + \epsilon(X).$$
- $T_1(X)$ is a particular case of distortion premium principle with concave distortion function
$$h(p) = \begin{cases} p(1 - \log p), & 0 < p \leq 1 \\ 0, & p = 0 \end{cases}$$
- We have
$$T_1(X) = \int_0^\infty E[X | X > t] \, dF(t) = \int_0^1 \mathbf{TVaR}_p(X) dp$$
- Drawback: The risk-loading is fixed!

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Incorporating risk-aversion into the risk-loading

- Consider a random sample $X_1, X_2, ..., X_n$ of independent claims with the same distribution F. Let $X_{1:n} \leq ... \leq X_{n:n}$ be the corresponding order statistics.

- Suppose that, in exchange for insuring the risk X, the insurer wishes to charge, at least, the expected payout of the risk $X_{i:n}$, for some $i = 1, ..., n$ (here, the choice of i reflects the degree of risk-aversion of the insurer).

- Under this assumption, a risk-adjusted tail distribution Y should satisfy the following properties under conditioning:

(P1') The conditional distribution $[Y | X_{i:n} = x]$ only depends on the tail of X.
(P2') $E[Y | X_{i:n} = x] \geq x$ for all x.
(P3') $[Y | X_{i:n} = x_1] \leq_{st} [Y | X_{i:n} = x_2]$ for $x_1 < x_2$.
(P4') If X is heavy tailed, the function $E[Y | X_{i:n} = x] - x$ increases as $x \to \infty$.

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Incorporating risk-aversion into the risk-loading

Theorem

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$$\left[Y_{X,i,n}\,|X_{i:n}=x\right] \overset{d}{=} [X\,|X>x] \text{ for all } x \geq 0,$$

where $X_{i:n}$ is the i-th order statistics of n independent copies of X, is a risk-adjusted distribution satisfying the properties P1', P2' and P3'. Moreover, if $\overline{F}$ is log-convex, then $Y_{X,i,n}$ also satisfies P4'.

Incorporating risk into the risk-loading

We study the premium principle

$$T_{i,n}(X) = E\left[Y_{X,i,n}\right] \quad \text{where} \quad \left[Y_{X,i,n} \,|\, X_{i:n} = x\right] \stackrel{d}{=} [X \,|\, X > x] \text{ for all } x \geq 0.$$

Theorem

For a risk X and $1 \leq i \leq n-1$:
(a) $Y_{X,i,n} \leq_{st} Y_{X,i+1,n}$
(b) $Y_{X,i,n} \leq_{st} Y_{X,i,n-1}$

The parameters i and n represent risk aversion when the other is fixed

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Different representations of $T_{i,n}(X)$

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Theorem

$T_{i,n}(X)$ is a distortion premium principle with concave distortion function

$$h_{i,n}(t) = 1 - c_{i,n} \int_t^1 \int_p^1 (1-u)^{i-1} u^{n-i-1} du dp, \; 1 \leq i \leq n.$$

where $c_{i,n} = \frac{n!}{(i-1)!(n-i)!}$.

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Theorem

$$T_{i,n}(X) = E[X] + \epsilon_{i,n}(X),$$

where the risk loading of the premium is given by

$$\epsilon_{i,n}(X) = \begin{cases} \frac{i}{n-i} E[X] - \frac{n}{n-i} \int_0^\infty \left(\int_0^{\overline{F}(x)} \beta_{n-i,i}(p)dp \right) dx, & 1 \leq i < n \\ n \sum_{j=n}^\infty \int_0^\infty \frac{\overline{F}(x)(1-\overline{F}(x))^j}{j} dx + E[X_{n:n} - X], & i = n \end{cases}$$

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Theorem

$$T_{i,n}(X) = \int_0^1 \mathbf{TVaR}_p(X) d\beta_{i,n-i+1}(p).$$

where

$$\beta_{i,j}(p) = \int_0^p \frac{(i+j-1)!}{(i-1)!\,(j-1)!} t^{i-1}(1-t)^{j-1}\,dt,\ 0 \leq p \leq 1,$$

is Pearson's incomplete beta function with parameters (i, j).

The case $T_{1,n}(X)$

Consider $T_{1,n}(X) = E\left[Y_{X,1,n}\right]$ where $\left[Y_{X,1,n} \,|X_{1:n} = x\right] \overset{d}{=} [X\,|X > x]$ for all $x \geq 0$.

- The risk loading, given by

$$\epsilon_{1,n}(X) = \frac{-1}{n-1}\int_0^{\infty}\left[\left(\overline{F}(x)\right)^n - \overline{F}(x)\right]dx.$$

is a decreasing function of n and $\epsilon_{1,n}(X) \to 0$ as n goes to infinite.

From the net premium to the CRE premium

$$E[X] \leq ... \leq T_{1,n+1}(X) \leq T_{1,n}(X) \leq ... \leq T_{1,1}(X) = T_1(X) = E[Y_X].$$

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- $T_{n,n}(X)$ is a distortion premium principle with distortion function

$$h_{n,n}(t) = nt \int_t^1 \frac{(1-u)^{n-1}}{u} du + 1 - (1-t)^n, \; 0 \leq t \leq 1.$$

Properties

- $h_{n,n}$ is strictly concave and $h'_{n,n}(0) = \infty$, which is a desirable property in reinsurance (Wang, 1996).

- Distortions satisfying this property are called “adapted” in Balbás et al. (2009). Another adapted distortion is the popular Wang Transform (Wang, 2000)

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$$T_1(X) = E\left[Y_X\right] = T_{1,1}(X) \leq T_{2,2}(X) \leq ... \leq T_{n,n}\left(X\right) \leq ..., \quad n \geq 2.$$

Upper bound for the n-risk loading

Let X be a risk with variance σ^2. Then, for $n \geq 1$,

$$\epsilon_{n,n}(X) \leq n\epsilon(X) + \frac{n-1}{\sqrt{2n-1}}\sigma.$$

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Uniform distribution $U \sim U(0, a)$

For $1 \leq i < n$:

$$\epsilon_{i,n}(U) =$$

$$\frac{ia}{2(n-i)} - ac_{i,n} \sum_{k=0}^{i-1} \binom{i-1}{k} \frac{(-1)^k}{(n-i+k)(n-i+k+1)(n-i+k+2)},$$

where $c_{i,n} = \frac{n!}{(i-1)!(n-i)!}$.

In particular:

$$T_{1,n}(U) = \frac{a(n+2)}{2(n+1)}, \qquad T_{n,n}(U) = \frac{a(2n+1)}{2(n+1)}.$$

Exponential distribution $Z \sim \text{Exp}(\lambda)$

For $1 \leq i < n$:

$$\epsilon_{i,n}(Z) =$$

$$\frac{i\lambda}{n-i} - \lambda c_{i,n} \sum_{k=0}^{i-1} \binom{i-1}{k} (-1)^k \frac{1}{(n-i+k)(n-i+k+1)^2}.$$

In particular:

$$T_{1,n}(Z) = \lambda + \frac{\lambda}{n}, \qquad T_{n,n}(Z) = \lambda \left(1 + \sum_{j=1}^{n} \frac{1}{j} \right).$$

Pareto distribution $W \sim \mathrm{P}(\alpha, \beta)$, $\alpha > 1$ and $\beta > 0$

For $1 \leq i < n$:

$$\epsilon_{i,n}(W) = \frac{i\beta}{(\alpha-1)(n-i)}$$

$$-\beta c_{i,n} \sum_{k=0}^{i-1} \binom{i-1}{k} \frac{(-1)^k}{(n-i+k)(n-i+k+1)(\alpha(n-i+k+1)-1)}$$

In particular:

$$T_{1,n}(W) = \frac{\beta}{n-1}\left[\frac{n}{\alpha-1} - \frac{1}{n\alpha-1}\right]$$

and

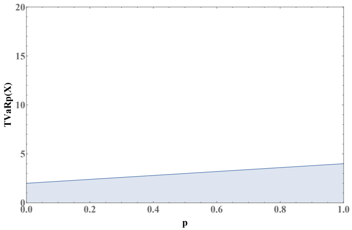
$$T_{n,n}(W) = -\beta + \frac{\beta\alpha}{\alpha-1}\frac{\Gamma(1-1/\alpha)\Gamma(n+1)}{\Gamma(n+1-1/\alpha)}.$$

A comparative study

- We compare the performance of the functionals considered in this paper by applying the premium principles to three risks with the same mean.

- Let U be a uniform $(0, 4)$, let Z be an exponential distribution of mean 2 and let W be a Pareto distribution with parameters $\alpha = 2$ and $\beta = 2$ (therefore, W has mean 2).

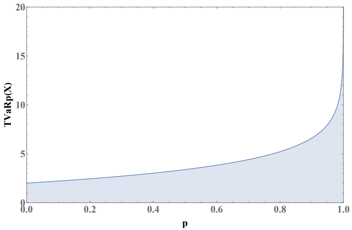
A comparative study



	n=1	n=2	n=5	n=10	n=20	n=50	n=100
i=1	3	2,66667	2,3333	2,18182	2,09524	2,03922	2,0198
i=2		3,33333	2,6666	2,36364	2,19048	2,07843	2,0396
i=5			3,6666	2,90909	2,47619	2,19608	2,09901
i=10				3,81818	2,95238	2,39216	2,19802
i=20					3,90476	2,78431	2,39604
i=50						3,96078	2,9901
i=100							3,9802

$$T_{i,n}(U), \quad U \sim U(0,4)$$

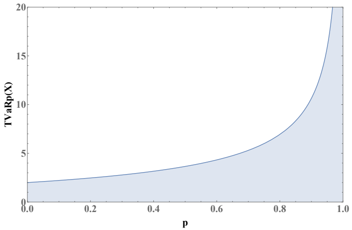
A comparative study



	n=1	n=2	n=5	n=10	n=20	n=50	n=100
i=1	4	3	2,4	2,2	2,1	2,04	2,02
i=2		5	2,9	2,42222	2,20526	2,08082	2,0402
i=5			6,5666	3,29127	2,55902	2,20851	2,10206
i=10				7,85794	3,33754	2,44132	2,20961
i=20					9,19548	3,00844	2,4438
i=50						10,9984	3,37634
i=100							12,3748

$$T_{i,n}(Z), \quad Z \sim Exp(1/2)$$

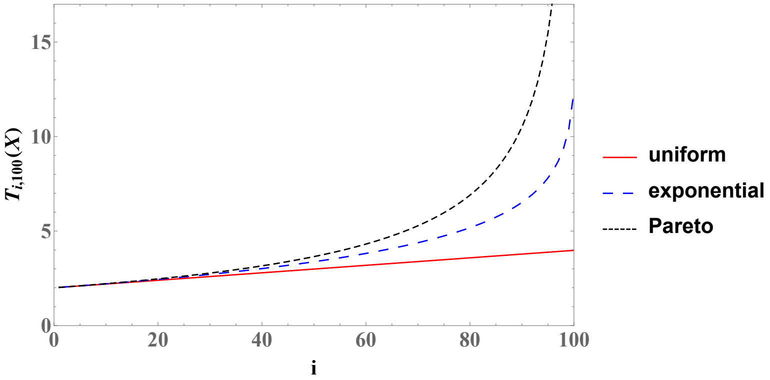
A comparative study



	n=1	n=2	n=5	n=10	n=20	n=50	n=100
i=1	6	3,33333	2,4444	2,21053	2,10256	2,0404	2,0201
i=2		8,66667	3,07937	2,4582	2,21344	2,08206	2,04051
i=5			14,254	3,58678	2,60919	2,2152	2,10364
i=10				20,7019	3,62163	2,46934	2,21578
i=20					29,9054	3,15538	2,47074
i=50						48,2581	3,64979
i=100							68,9868

$$T_{i,n}(W), \quad W \sim Pa(2,2)$$

A comparative study



Conclusions

- Given a distribution X, we have introduced, from three different approaches, a family of premium principles $\{T_{i,n}(X)\}$ with parameters $n \geq 1$ and $1 \leq i \leq n$ that reflect the degree of risk-aversion of the insurer.
- Firstly, $T_{i,n}(X)$ has been derived as the expected value of a risk-adjusted distribution that is well-behaved under conditioning with respect to X.
- Secondly, $T_{i,n}(X)$ has been presented as a particular distortion premium principle with a concave distortion.
- Finally, $T_{i,n}(X)$ has been geometrically interpreted as a weighted area under the curve defined by $\mathbf{TVaR}_p(X)$, where the weights reflect the attitude toward risk of the insurer.
- We think that $T_{i,n}(X)$ may play an interesting role in actuarial science.

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