Dynamic Frailty Count Process in Insurance: Estimation, Pricing and Forecasting

Yang Lu

Aix-Marseille University, France

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INTRODUCTION

Outline

- We propose a stochastic intensity count process model for non-life insurance.
- That is, the individual risk factor is unobservable and time-varying.
- Its dynamics is autoregressive gamma (ARG).
- The model allows for closed form Bayes premium, and provides a unified framework for estimation and forecasting.

The credibility literature

- In (say, car) insurance, a policyholder usually has a claim history of counts over several years.
- The actuary updates the premium after each year.
- Ideally he/she should use the Bayes rule.
- However, the Bayes premium is not always tractable.
- Credibility theory proposes to use *linear* functions of past counts to predict future count.

However, the credibility approach has some limits.

1. Approximation error.
2. Not adapted for non-linear pricing (Esscher transform, etc.)
3. Not adapted for forecasting, and risk management (heteroscedasticity).

An exception: Dionne and Vanasse [1989] introduces the static Poisson-Gamma model.

The Poisson-Gamma model

N_t: the claim count at period t. It is Poisson $\mathcal{P}(\lambda_t U)$, where

- λ_t is the a priori intensity: depends on your age, your car,...
- U is the individual, residual heterogeneity.

If U is gamma distributed $\Gamma(\delta, 1/\delta)$, then the posterior premium has a simple form:

$$\mathbb{E}[N_{T+1}|N_1, N_2, ..., N_T] = \frac{\delta + \sum_t N_t}{\delta + \sum_t \lambda_t}.$$

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A dynamic, gamma frailty model

- However, when U is static, previous claims have the same weight.
- We propose to replace U by (U_t), see also Pinquet et al. [2001], Bolancé et al. [2003], Brouhns et al. [2003].
- “stochastic intensity", or “dynamic frailty".
- Interpretation: changing habit, or *individual effort*.
- Up to now only credibility premium has been obtained for such models.
- We show that when (U_t) is autoregressive gamma, we get exact Bayes premium.
- This allows us to assess the pricing bias of the credibility premium.

AUTOREGRESSIVE GAMMA PROCESS

Autoregressive gamma (ARG) process

A Markov process with Gamma marginal $\Gamma(\delta, \frac{c}{1-\rho})$, with a conditional Laplace transform:

$$\mathbb{E}[\exp(-sU_{t+1}) \mid U_t] = \frac{1}{(1+cs)^\delta} \exp\Big(- \frac{\rho s}{1+cs} U_t \Big).$$

See Gouriéroux and Jasiak [2006] for details.

Its background

- An example: the square of an AR(1) Gaussian process is autoregressive gamma.
- Discrete time counterpart of the Cox-Ingersoll-Ross process.
- Affine process (the log of conditional Laplace transform is affine).
- A dynamic Poisson-Gamma representation:

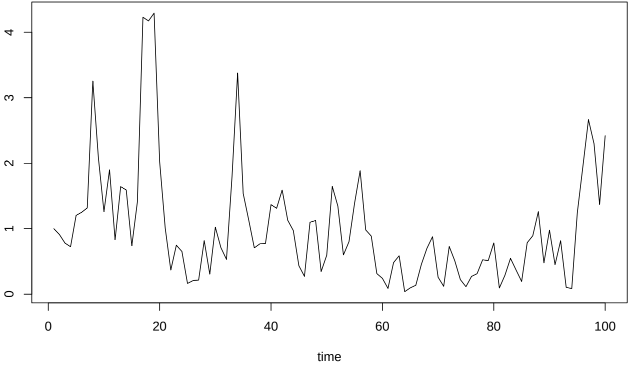
$$Z_t \sim \mathcal{P}\left(\frac{\rho}{c} U_t\right)$$
$$U_{t+1} \sim \Gamma(\delta + Z_t, c)$$

where Z_t is unobservable: “switching regime".

The state-space representation of the claims count (N_t):

$$
\begin{array}{ccccccc}
\dots & U_{t-1} \to Z_{t-1} & \to & U_t \to Z_t & \to & U_{t+1} \to Z_{t+1} & \dots \\
 & & & & & & \\
\dots & \Big\downarrow & & \Big\downarrow & & \Big\downarrow & \dots \\
\dots & N_{t-1} & & N_t & & N_{t+1} & \dots
\end{array}
$$

Simulated path of an ARG process



The (weak) linear representation

(U_t) has a *weak* AR(1) representation:

$$\mathbb{E}[U_{t+1}|U_t] = c\delta + \rho U_t.$$

- Stationary if $\rho < 1$.
- Autocorrelation:

$$\text{corr}(U_t, U_{t+h}) = \rho^h, \qquad \forall h$$

- If $\rho = 1$, $U_t = U_{t+1}$, we get the static gamma distribution.

Forecasting the future count

Assume: $N_t = \mathcal{P}(\lambda U_t), \forall t$.
The Bayes premium for period 2 is:

$$\mathbb{E}[N_2 \mid N_1] = \lambda_2 \mathbb{E}[U_2 \mid N_1] = \lambda_2 \frac{\mathbb{E}[U_2 U_1^{N_1} e^{-\lambda_1 U_1}]}{\mathbb{E}[U_1^{N_1} e^{-\lambda_1 U_1}]}.$$

This latter involves the joint Laplace transform:

$$\mathbb{E}[U_2 U_1^{N_1} e^{-\lambda_1 U_1}] = (-1)^{n_1+1} \frac{\partial^{n_1+1}}{\partial_1 s_1^{n_1} \partial s_2} \mathbb{E}[e^{-s_1 U_1 - s_2 U_2}] \Big|_{(\lambda_1, 0)}$$

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Joint Laplace transform

Thus we need a tractable expression for the joint Laplace transform.

Since $\mathbb{E}[e^{-s_2 U_2} \mid U_1]$ and $\mathbb{E}[e^{-s_1 U_1}]$ are tractable, we get:

Proposition 1

$$\mathbb{E}[e^{-s_1 U_1 - s_2 U_2}] = \frac{1}{(1 + \tilde{c} s_1 + \tilde{c} s_2 + \tilde{c}^2 (1-\rho) s_1 s_2)^{\delta}},$$

where $\tilde{c} = \frac{c}{1-\rho}$.

Thus we have an *exact* pricing formula for ARG-based model, for $T = 2$.

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The general case

More generally, the Bayes premium

$$\mathbb{E}[N_{T+1} \mid N_1, N_2, ..., N_T]$$

involves the joint Laplace transform of $(U_1, U_2, ..., U_T)$, which has a closed form expression under ARG:

Proposition 2

$$\mathbb{E}[e^{-s_1 U_1 - ... - s_T U_T}] = \frac{1}{\left\{ \det \left[I + \frac{c}{1-\rho} R \; Diag(s_1, s_2, ..., s_T) \right] \right\}^{\delta}},$$

where $R = (\rho^{|i-j|/2})_{1 \leq i,j \leq T}$.

ASSESSING THE BIAS OF THE CREDIBILITY PREMIUM

For different claim histories, we compare the future premium given by two methods, under the ARG model:

- The credibility premium:

$$p_1(N_1, ..., N_T) = p_0 + \sum_{h=1}^{T} a_h N_{T+1-h},$$

  where a_h are regression coefficients.

- The Bayes premium:

$$p_2(N_1, ..., N_T) = \mathbb{E}[N_{T+1} | N_1, ..., N_T].$$

Such an analysis is rarely done in the literature, due to the difficulty of deriving the Bayes premium.

$T = 2$

	Bayes	Credibility
$p_3(0,0)$	0.89 λ	0.90 λ
$p_3(0,1)$	1.75 λ	1.76 λ
$p_3(1,0)$	1.49 λ	1.50 λ
$p_3(1,1)$	2.47 λ	2.37 λ
$p_3(0,2)$	2.60 λ	2.63 λ
$p_3(2,0)$	2.08 λ	2.11 λ
$p_3(2,1)$	3.10 λ	2.98 λ
$p_3(1,2)$	3.36 λ	3.24 λ

TablePremium for the third period as a function of N_1 and N_2.

A convexity analysis

- The credibility model is based on a linear function of the previous count. That is:

$$p_3(x,y) = p_3(0,0) + \Delta_x p_3(0,0)x + \Delta_y p_3(0,0)y, \qquad \forall x, y \in \mathbb{N},$$

where

$$\Delta_x p_3(x,y) = p_3(x+1,y) - p_3(x,y) = \text{Cst}_1,$$
$$\Delta_y p_3(x,y) = p_3(x,y+1) - p_3(x,y) = \text{Cst}_2.$$

Thus:

$$\Delta_{xx} = \Delta_x(\Delta_x p_3) = 0,$$
$$\Delta_{yy} = \Delta_y(\Delta_y p_3) = 0,$$
$$\Delta_{xy} = \Delta_x(\Delta_y p_3) = 0.$$

As a comparison, for the Bayes premium, we have:

$\Delta^2_{xx} p_3(0,0)$	-0.01	$\Delta^2_{yy} p_3(0,0)$	-0.01	$\Delta^2_{xy} p_3(0,0)$	0.12
$\Delta^2_{xx} p_3(0,1)$	-0.09	$\Delta^2_{yy} p_3(0,1)$	0.01	$\Delta^2_{xy} p_3(0,1)$	0.04
$\Delta^2_{xx} p_3(1,0)$	0	$\Delta^2_{yy} p_3(1,0)$	-0.09	$\Delta^2_{xy} p_3(1,0)$	0.04

Thus

$$\begin{aligned} p_3(x,y) = p_3(0,0) &+ \Delta_x p_3(0,0)x + \Delta_y p_3(0,0)y \\ &+ \Delta^2_{xy} p_3(0,0)\frac{xy}{2} + \Delta^2_{xx} p_3(0,0)\frac{x^2}{2} + \Delta^2_{yy} p_3(0,0)\frac{y^2}{2} \\ &+ \text{ higher order terms} \end{aligned}$$

To summarize,

- the two premia differ in terms of the convexity adjustment for large past counts,
- this explains large pricing bias of the credibility premium, when N_1, or/and N_2 is large,
- roughly, the Bayes premium is higher (resp. lower) for clients with many (resp. few) previous accidents.

CONCLUSION

- We have introduced the ARG-based count process model to the Insurance literature.
- Our model inherits some nice features of the static Poisson-Gamma model.
- For the first time, exact Bayes premium is derived for a dynamic model, and the bias of credibility premium is assessed.
- Finally, a unified framework for estimation, pricing and forecasting since:

Other advantages

- Estimation: efficient estimation methods exist (composite likelihood).
- (Nonlinear) pricing: explicit Esscher transform based premium:

$$p_{3,T+1}(N_1, ..., N_T) := \frac{\mathbb{E}[N_{T+1} e^{\alpha N_{T+1}} \mid N_1, ..., N_T]}{\mathbb{E}[e^{\alpha N_{T+1}} \mid N_1, ..., N_T]}.$$

- Forecasting: captures heteroscedasticity: the larger $\mathbb{E}[N_{T+1}|N_T, ..., N_1]$, the larger $\mathbb{V}[N_{T+1}|N_T, ..., N_1]$.

THANKS FOR YOUR ATTENTION.

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