Analysing systematic risk and diversification benefits

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8 September 2016
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Contents of this presentation are mainly based on three papers:


Motivation

► Diversification benefits are typical when modelling the aggregate risk of imperfectly dependent random component losses.

► The total diversification benefit is allocated back to component losses, e.g. for performance measurement.

► Key questions usually are:
  ► How I explain the allocated diversification benefits?
  ► Which “part” of the joint distribution is contributing to the benefits?
  ► How does the dependence structure affect the benefits?

► A formal approach is proposed to answer the above questions.

► An illustration is used to show the effectiveness of the proposed approach.
Standalone risk

► Consider a random component loss $x > 0$ with distribution function $F$ and percentile rank $u = F(x)$ which is uniformly distributed on $(0,1)$.

► The standalone risk of a loss $x > 0$ is measured by distortion:

$$r = \int_0^\infty 1 - \Phi\{F(x)\}dx - E(x) = \text{cov}\{x, \phi(u)\}$$

► In the above $\Phi$ is a distortion function and $\phi = \Phi'$ is an aversion function.

► $\Phi$ is convex and $\phi$ is non-decreasing.

► Common examples of distortion risks are the conditional-tail-expectation or expected shortfall, expected-maximal-loss and proportional hazards risk.

► Distortion risks are coherent and most importantly, sub-additive.
Suppose $x$ is aggregated with several component losses, e.g. losses from multiple insurance classes or business units.

The total loss is $x_+$. The distribution function of $x_+$ is $F_+$ and $u_+ = F_+(x_+)$. Then the aggregate risk, again using the distortion approach, is

$$r_+ = \text{cov}\{x_+, \phi(u_+)\}$$

The systematic risk of $x$, using the well known Euler principle, is given by

$$\bar{r} = \text{cov}\{x, \phi(u_+)\}$$

The systematic risk of $x$ is also the portion of $r_+$ allocated to $x$.

Adding up systematic risks across all component losses gives $r_+$. 
Diversification benefits

► The Euler principle gives several desirable practical properties, in particular non-negative allocated diversification benefits for all component losses.

► Therefore the standalone risk $r$ of $x$ can be split into systematic risk and a diversification benefit which are both non-negative:

$$\bar{r}, \quad r - \bar{r} \geq 0$$

► Key focus is on the systematic risk ratio:

$$\theta = \frac{\bar{r}}{r} = \frac{\text{cov}\{x, \phi(u+)\}}{\text{cov}\{x, \phi(u)\}} \leq 1$$

► A systematic risk ratio $\theta = 1$ indicates zero diversification benefit for $x$.

► As $\theta$ decreases, the diversification benefit for $x$ increases and vice versa.

► $\theta$ decreases when $x$ becomes more weakly dependent on $x_+$ (intuitive).
It is well known that any $x > 0$ can be decomposed into infinitesimal layers:

$$x = \int_0^1 I(x > k) dk$$

Here $I(x > k) dk$ is the $k$-layer of $x$, or the portion $[k, k+dk]$ of $x$.

Layers are common quantity in excess-of-loss reinsurance and asset-backed securities (called tranches instead of layers).

By substituting $k = F^{-1}(\alpha) = V(\alpha)$ where $V(\alpha)$ is the Value-at-Risk or VaR of $x$ at threshold $\alpha$:

$$x = \int_0^1 I(u > \alpha)V'(\alpha) d\alpha$$

Now $I(u > \alpha)V'(\alpha) d\alpha$ is the $\alpha$-VaR layer of $x$.

Now loss $x$ is decomposed into infinitesimal VaR layers.
Standalone and systematic risk of VaR layers

Using the VaR-layer decomposition of $x$, we can rewrite the standalone and systematic risks of $x$ as:

$$r = \int_0^1 \text{cov} \{I(u > \alpha)V'(a), \phi(u)\} d\alpha, \quad \bar{r} = \int_0^1 \text{cov} \{I(u > \alpha)V'(a), \phi(u_+)\} d\alpha$$

Hence the standalone risk density of $x$ at the $\alpha$–VaR layer is:

$$r_\alpha = V'(a) \text{cov} \{I(u > \alpha), \phi(u)\}$$

And the systematic risk density of $x$ at the $\alpha$–VaR layer is:

$$\bar{r}_\alpha = V'(a) \text{cov} \{I(u > \alpha), \phi(u_+)\}$$

The diversification benefit of the $\alpha$–VaR layer of $x$ is the difference between the two risk densities.
Systematic risk ratio of VaR layers

► The systematic risk ratio of the $\alpha$–VaR layer of $x$ is:

$$
\theta_\alpha = \frac{V'(a) \text{cov}\{I(u > \alpha), \phi(u_+)\}}{V'(a) \text{cov}\{I(u > \alpha), \phi(u)\}} = \frac{\text{cov}\{I(u > \alpha), \phi(u_+)\}}{\text{cov}\{I(u > \alpha), \phi(u)\}} \leq 1
$$

► $\theta_\alpha$ measures the portion of the risk of the $\alpha$–VaR layer which is remaining after diversification with other component losses.

► There is zero diversification between VaR layers of $x$ as these layers are all comonotonic, and distortion risks are additive under comonotonicity.

► Therefore the diversification at the $\alpha$–VaR layer of $x$ is negatively related to its dependence with $x_+$ (after application of $F_+$ and $\phi$).

► This dependence is also the local dependence between $x$ at $\alpha$–VaR and $u_+$. 
Systematic risk ratio of VaR layers

► The overall systematic risk ratio of $x$ can be rewritten as:

$$\theta = \frac{-\bar{r}}{r} = \frac{\int_0^1 r_\alpha \theta_\alpha \, d\alpha}{\int_0^1 r_\alpha \, d\alpha}$$

► Or a weighted average of systematic risk ratios of VaR layers of $x$.

► Hence the overall extent of diversification is a weighted average of the extent of diversification in individual VaR layers, which in turn is linked to local dependence between losses.

► The extent of diversification in VaR layers is driven by local dependence.
Recap

Quick recap before the illustration:

► We have adopted popular measures of standalone and systematic risks
► We want to analyse diversification benefits allocated to component losses
► A loss can be decomposed into infinitesimal VaR layers
► We can compute standalone, systematic and diversified risks of a VaR layer
► The decomposition allows us to granularly analyse risk and diversification
► The overall diversification is tied to a weighted average of local dependence
Illustration – setup

► Suppose $x$ is the insurance loss on the personal property class and $y$ is the insurance loss on the commercial property class.

► Both $x$ and $y$ comprise of an independent working loss and a comonotonic natural catastrophe loss (e.g. earthquake).

► Write:

$$x = w_x + c_x, \quad y = w_y + c_y$$

► $w_x$ and $w_y$ are independent working losses

► $c_x$ and $c_y$ are comonotonic natural catastrophe losses

► Assume working and catastrophe losses are independent.
Loss distributions

► All amounts are in millions of Euros
► \( x \sim N(400,100^2) \) and \( y \sim N(600,200^2) \)
► \( c_x = 0.4c \) and \( c_y = 0.6c \) where \( c \) is the total catastrophe loss
► \( c \) has a Fréchet heavy tailed distribution with \( \mu = 300, \sigma = 100, \zeta = 0.1 \)
Risk measurement

- Let’s use the following distortion function at $t=0.75$:

$$
\Phi(u) = I(u > t) \frac{u - t}{1 - t}, \quad \phi(u) = \frac{I(u > t)}{1 - t}
$$

- This gives the conditional-tail-expectation (or expected shortfall) measure of risk. The standalone and systematic risks of $x$ are:

$$
r = E\{x \mid x > F^{-1}(t)\} - E(x), \quad \overline{r} = E\{x \mid x_+ > F_+^{-1}(t)\} - E(x)
$$

- Standalone risk of $x$: expected value of $x$ given it exceeds its 75%-VaR.
- Systematic risk of $x$: conditional mean of $x$ given $x_+$ exceeds its 75%-VaR.
- In both cases we need to subtract the mean of $x$ to calculate risk.
Copula plot of $x$ and $y$

- The copula scatterplot is mostly independent
- There is upper tail dependence due to the cat losses
- Pearson’s correlation between $x$ and $y$ is $0.37$
- Spearman’s rank correlation is $0.31$
- This suggests weak overall dependence between $x$ and $y$
- However diversification benefits are small – see next
Standalone and systematic risks

- For loss \( x \) there is a 28\% risk diversification:
  \[ r = 174, \quad \bar{r} = 125, \quad \theta = 0.72 \]

- For loss \( y \) there is an 8\% risk diversification:
  \[ r = 310, \quad \bar{r} = 286, \quad \theta = 0.92 \]

- The overall diversification is 15\% although \( x \) and \( y \) are weakly correlated.

- Therefore we need to analyse diversification benefits across VaR layers to understand why this is the case.
Risks across VaR layers of $x$

Green line is the plot of $\theta_\alpha$ against $\alpha$.

Red and blue lines are standalone and systematic risks of $\alpha$–VaR layers of $x$

Area under the curves are the overall risks and diversification benefits.
Risks across VaR layers of y

Similar diagrams as previous slide, but for y.

Stronger dependence with the total loss especially in the tail.

Stronger local dependence leads to smaller diversification benefit for y.
Key observations

► Using the CTE @ 75% as the risk measure, standalone risk is concentrated in the higher VaR layers of x and y as both losses are right skewed.
► Standalone risks peak at the 75% VaR layers of x and y, due to the selected risk measure.
► For x, there is a reasonable degree of diversification except for the highest VaR layers due to tail dependence.
► For y, diversification is significantly lower due to stronger dependence with the total loss. Diversification again disappears for the highest VaR layers.
► y has stronger dependence with the total loss as a result of more volatile working and catastrophe losses.
Alternative catastrophe assumptions

- Suppose instead $c_x$ and $c_y$ are independent
- Marginal distributions of $x$ and $y$ are unchanged
- However now $x$ and $y$ are independent
- Standalone risks are unchanged
- The diversification benefit for $x$ increases from 28% to 51%
- The diversification benefit for $y$ increases from 8% to 13%
- Overall diversification is 27% compared to 15% previously
Risks across VaR layers of $x$
Risks across VaR layers of $y$
Comonotonic and independent cat losses

Standalone and systematic risks of VaR layers of $x$ with comonotonic (left) and independent (right) cat losses.
Comonotonic and independent cat losses

Standalone and systematic risks of VaR layers of $y$ with comonotonic (left) and independent (right) cat losses.
Financial markets example

- Financial markets are known to have strong tail dependence
- Tail volatility is also typically high
- This leads to low diversification despite weak overall correlation since:
  - Risks are concentrated in higher VaR layers due to tail volatility
  - Systematic risk ratios are one in high VaR layers due to tail dependence
  - As a result overall diversification is low
End