Asymptotic Ruin Probabilities for a Multidimensional Renewal Risk Model with Multivariate Regularly Varying Claims

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1. Introduction.

Consider an insurance company which simultaneously operates d kinds of businesses. Its surplus process can be described by the following multidimensional risk model:

$$\begin{pmatrix} U_1(t) \\ \vdots \\ U_d(t) \end{pmatrix} = \begin{pmatrix} \rho_1 x \mathrm{e}^{rt} \\ \vdots \\ \rho_d x \mathrm{e}^{rt} \end{pmatrix} + \begin{pmatrix} c_1 \int_0^t \mathrm{e}^{r(t-s)} \mathrm{d}s \\ \vdots \\ c_d \int_0^t \mathrm{e}^{r(t-s)} \mathrm{d}s \end{pmatrix} - \begin{pmatrix} \sum_{i=1}^{N_1(t)} X_{1i} \mathrm{e}^{r(t-\tau_{1i})} \\ \vdots \\ \sum_{i=1}^{N_d(t)} X_{di} \mathrm{e}^{r(t-\tau_{di})} \end{pmatrix}, \tag{1}$$

where $\{(U_1(t), \ldots, U_d(t))\,; t \geq 0\}$ denotes the multidimensional surplus process, $r \geq 0$ the constant force of interest, $(\rho_1 x, \ldots, \rho_d x)$ the vector of initial surpluses assigned to different businesses with positive ρ_1 , . . . , ρ_d such that $\sum_{k=1}^{d} \rho_k = 1$, $(c_1, \ldots, c_d)$ the vector of constant premium rates, $\{(X_{1i}, \ldots, X_{di}); i \geq 1\}$ the sequence of claim-size vectors, and τ_{k1}, τ_{k2}, . . . the claim-arrival times of the kth business with the corresponding claim-number process $\{N_k(t); t \geq 0\}$ for $k = 1, \ldots, d$.

Define the finite-time and infinite-time ruin probabilities corresponding to risk model (1) as

$$\psi\left(x;T\right)=\mathbb{P}\left(T_{\max}\leq T\right|\left(U_1(0),\ldots,U_d(0)\right)=\left(\rho_1 x,\ldots,\rho_d x\right)\right),$$

and

$$\psi\left(x\right)=\mathbb{P}\left(T_{\max}<\infty\right|\left(U_1(0),\ldots,U_d(0)\right)=\left(\rho_1 x,\ldots,\rho_d x\right)\right),$$

where

$$T_{\max}=\inf\left\{t>0:\max\left\{U_1(t),\ldots,U_d(t)\right\}<0\right\}$$

denotes the ruin time with $\inf\emptyset=\infty$ by convention.

The present paper is devoted to extend the existing works from three main aspects. First, we drop the restriction that all businesses have a totally identical claim-number process, and instead introduce a general dependence structure into the different claim-number processes. Second, we synthetically model the dependence and heavy-tailed nature of claim sizes by a unified framework of multivariate regular variation. Last but not the least, we obtain asymptotic expansions for both finite-time and infinite-time ruin probabilities.

2. Preliminaries.

A distribution function $F = 1 - \overline{F}$ on $[0, \infty)$ is said to belong to the class of regular variation if $\overline{F}(x) > 0$ for all $x \geq 0$ and the relation

$$\lim_{x \to \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\alpha}, \qquad y > 0, \tag{2}$$

holds for some $0 < \alpha < \infty$. We signify the regularity property in (2) as $\overline{F} \in \mathcal{R}_{-\alpha}$.

For a distribution function F with $\overline{F} \in \mathcal{R}_{-\alpha}$ for some $0 < \alpha < \infty$, we know that, for any $0 < p_1 < \alpha < p_2 < \infty$ and $C > 1$, there is some $D > 0$ such that the inequalities

$$\frac{1}{C} \min\left\{y^{-p_1}, y^{-p_2}\right\} \leq \frac{\overline{F}(xy)}{\overline{F}(x)} \leq C \max\left\{y^{-p_1}, y^{-p_2}\right\} \tag{3}$$

hold whenever $x > D < xy$. We can derive from (3) that if $\overline{F} \in \mathcal{R}_{-\alpha}$ then, for any $p > \alpha$,

$$x^{-p} = o\left(\overline{F}(x)\right), \qquad x \to \infty. \tag{4}$$

It is known that a distribution function F with a regularly varying tail belongs to the long-tailed distribution class $\mathcal{L}$ characterized by $\overline{F}(x) > 0$ for all $x \geq 0$ and the relation

$$\lim_{x \to \infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} = 1, \qquad y \in (-\infty, \infty).$$

Consider the multidimensional risk model (1). Throughout, we assume that $\{(N_1(t), \ldots, N_d(t)); t \geq 0\}$ and $\{(X_{1i}, \ldots, X_{di}); i \geq 1\}$ are mutually independent. For $k = 1, \ldots, d$, denote by $\theta_{k1} = \tau_{k1}$ and $\theta_{ki} = \tau_{ki} - \tau_{k,i-1}$ for $i = 2, 3, \ldots$ the inter-arrival times of claims from the kth business. We introduce a general dependence structure into the different claim-number processes through the following assumption:

Assumption 2.1. $\{(\theta_1, \ldots, \theta_d), (\theta_{1i}, \ldots, \theta_{di}); i \geq 1\}$ is a sequence of independent and identically distributed (i.i.d.) nonnegative random vectors, but the d components of each vector can be arbitrarily dependent.

Clearly, under Assumption 2.1 all of $\{N_1(t); t \geq 0\}, \ldots, \{N_d(t); t \geq 0\}$ are traditional renewal processes, and they inherit dependences from that among $\theta_1, \ldots, \theta_d$. Further, for $(t_1, \ldots, t_d) \in \mathbb{R}_+^d$, we write

$$N(t_1, \ldots, t_d) = \max\{i : \tau_{1i} \leq t_1, \ldots, \tau_{di} \leq t_d\} = \min_{1 \leq k \leq d} \{N_k(t_k)\} \tag{5}$$

and

$$\lambda(t_1, \ldots, t_d) = \mathbb{E}(N(t_1, \ldots, t_d)) = \sum_{i=1}^{\infty} \mathbb{P}(\tau_{1i} \leq t_1, \ldots, \tau_{di} \leq t_d), \tag{6}$$

which are called in the literature the d-dimensional renewal process and the corresponding renewal function respectively.

Before modeling the dependence structure among the claim sizes, we need to introduce the concept of multivariate regular variation (MRV) first. A random vector $(Z_1, \ldots, Z_d)$ taking values in $[0, \infty]^d \setminus \{\mathbf{0}\}$ is said to follow a distribution with a multivariate regularly varying tail if there exist some $0 < \alpha < \infty$, some distribution function F with $\overline{F} \in \mathcal{R}_{-\alpha}$, and some Radon measure ν on $[0, \infty]^d \setminus \{\mathbf{0}\}$ satisfying $\nu\left([0, \infty]^d \setminus \{\mathbf{0}\}\right) > 0$ such that the following vague convergence holds as $x \to \infty$:

$$\frac{1}{\overline{F}(x)} \mathbb{P}\left(\frac{(Z_1, \ldots, Z_d)}{x} \in \cdot\right) \xrightarrow{v} \nu(\cdot) \quad \text{on } [0, \infty]^d \setminus \{\mathbf{0}\}.$$

In this case, we write $(Z_1, \ldots, Z_d) \in \mathrm{MRV}(\alpha, F, \nu)$.
Now we model the dependences among the claim sizes from different businesses via the framework of MRV. Concretely speaking, we assume:
Assumption 2.2. $\{(X_1, \ldots, X_d), (X_{1i}, \ldots, X_{di}); i \geq 1\}$ is a sequence of i.i.d. nonnegative random vectors with $(X_1, \ldots, X_d) \in \mathrm{MRV}(\alpha, F, \nu)$ such that $\nu((\mathbf{1}, \infty]) > 0$.

We can derive from Assumption 2.2 that

$$\nu(sK) = s^{-\alpha}\nu(K) \text{ for } s \in (0,\infty) \text{ and Borel set } K \subset [0,\infty]^d \setminus \{\mathbf{0}\}, \tag{7}$$

$$\lim_{x\to\infty} \frac{1}{\overline{F}(x)} \mathbb{P}\left(\bigcap_{k=1}^{d} \{X_k > x\}\right) = \nu\left((\mathbf{1},\infty]\right) > 0, \tag{8}$$

and

$$\lim_{x\to\infty} \frac{\mathbb{P}\left(X_k > x\right)}{\overline{F}(x)} = \nu\left((\mathbf{1}_k,\infty]\right) =: a_k > 0, \quad k = 1,\ldots,d, \tag{9}$$

where $\mathbf{1}_k$ is the vector with the kth element being 1 and the other elements being 0.

Relation (9) indicates that the tails of $X_1, \ldots, X_d$ are regularly varying and mutually comparable. This fact, combined with (8), implies that $X_1, \ldots, X_d$ are pairwise asymptotically dependent. Additionally, we know from (7) and (8) that

$$\lim_{x \to \infty} \frac{1}{\overline{F}(x)} \mathbb{P}\left(\bigcap_{k=1}^{d} \{X_k > b_k x\}\right) = \nu\left((b_1, \infty] \times \cdots \times (b_d, \infty]\right) > 0 \tag{10}$$

holds for any $(b_1, \ldots, b_d) \in [0, \infty]^d \setminus \{\mathbf{0}\}$.
In what follows, for any $(b_1, \ldots, b_d) \in [0, \infty]^d \setminus \{\mathbf{0}\}$, we will write

$$\nu\left((b_1, \infty] \times \cdots \times (b_d, \infty]\right) =: V\left(b_1, \ldots, b_d\right). \tag{11}$$

3. Main Results.

Theorem 1. *Consider risk model (1). Let Assumptions 2.1 and 2.2 hold. Then, for every T such that $\lambda(T, \ldots, T) > 0$, we have*

$$\psi(x, T) \sim \left[\int_{0-}^{T} \cdots \int_{0-}^{T} V\left(\rho_1 \mathrm{e}^{rt_1}, \ldots, \rho_d \mathrm{e}^{rt_d}\right) \lambda\left(\mathrm{d}t_1, \ldots, \mathrm{d}t_d\right) \right] \overline{F}(x). \qquad (12)$$

Next, we focus on the infinite-time ruin probability. To this end, we naturally require $r > 0$ in risk model (1) for the convergence of quantities under consideration.

Theorem 2. *In addition to the other conditions of Theorem 1, if* $r > 0$ *then (12) holds also for* $T = \infty$, *i.e., we have*

$$\psi(x) \sim \left[\int_{0-}^{\infty} \cdots \int_{0-}^{\infty} V\left(\rho_1 \mathrm{e}^{rt_1}, \ldots, \rho_d \mathrm{e}^{rt_d}\right) \lambda(\mathrm{d}t_1, \ldots, \mathrm{d}t_d) \right] \overline{F}(x) . \qquad (13)$$

Particularly, if the businesses share a common claim-number process, i.e., $N_1(t) \equiv \cdots \equiv N_d(t) \equiv N(t)$, then we can immediately obtain more elegant and transparent forms for (12) and (13) by applying (7).

Corollary 1. *In addition to the other conditions of Theorem 1, if* $N_1(t) \equiv \cdots \equiv N_d(t) \equiv N(t)$ *with* $\lambda(t) = \mathbb{E}(N(t))$ *then*

$$\psi(x,T) \sim \left(\int_{0-}^{T} \mathrm{e}^{-\alpha r t} \lambda(\mathrm{d}t)\right) V(\rho_1, \ldots, \rho_d)\, \overline{F}(x). \tag{14}$$

Further if $r > 0$ *then*

$$\psi(x) \sim \frac{\mathbb{E}\left(\mathrm{e}^{-\alpha r \theta}\right)}{1 - \mathbb{E}\left(\mathrm{e}^{-\alpha r \theta}\right)} V(\rho_1, \ldots, \rho_d)\, \overline{F}(x), \tag{15}$$

where θ *is the generic random variable of the inter-arrival times of* $\{N(t); t \geq 0\}$*.*

Formulas (12)–(15) reveal that the ruin probabilities of risk model (1) with our dependence structures assume a form of some constant times $\overline{F}(x)$. Although the constants before $\overline{F}(x)$ are involved in general, the formulas enable us to easily conduct numerical estimates for the ruin probabilities;

4. Lemmas.

Lemma 2. *Let random vector* $(Z_1, \ldots, Z_d) \in \mathrm{MRV}(\alpha, F, \nu)$*, and let* $(\xi_1, \ldots, \xi_d)$ *be a nonnegative random vector with arbitrarily dependent components satisfying* $\mathbb{E}(\xi_k^p) < \infty$ *for some* $p > \alpha$ *and all* $1 \leq k \leq d$*. Assume that* $(\xi_1, \ldots, \xi_d)$ *and* $(Z_1, \ldots, Z_d)$ *are independent. Then, for every Borel set* $K \subset [0, \infty]^d \setminus \{\mathbf{0}\}$*, we have*

$$\lim_{x \to \infty} \frac{1}{\overline{F}(x)} \mathbb{P}\left(\frac{(\xi_1 Z_1, \ldots, \xi_d Z_d)}{x} \in K \right) = \mathbb{E}\left[\nu\left(\xi^{-1} K \right) \right], \tag{16}$$

where $\xi^{-1} K = \{(b_1, \ldots, b_d) | (\xi_1 b_1, \ldots, \xi_d b_d) \in K\}$*.*

Lemma 3. *Let the conditions of Theorem 1 hold, and let* $\{\xi_{ki}; 1 \leq k \leq d, i \geq 1\}$ *be a sequence of nonnegative and arbitrarily dependent random variables satisfying* $\mathbb{E}\left(\xi_{ki}^{p}\right) < \infty$ *for some* $p > \alpha$ *and all* $1 \leq k \leq d$, $i \geq 1$. *Assume that* $\{\xi_{ki}; 1 \leq k \leq d, i \geq 1\}$ *and* $\{(X_{1i}, \ldots, X_{di}); i \geq 1\}$ *are mutually independent. Then, for any* $n_1 \geq 1$, $\ldots$, $n_d \geq 1$ *with* $\hat{n} = \min_{1 \leq k \leq d}\{n_k\}$, *we have*

$$\mathbb{P}\left(\bigcap_{k=1}^{d}\left\{\sum_{i=1}^{n_k}\xi_{ki}X_{ki} > \rho_k x\right\}\right) \sim \sum_{i=1}^{\hat{n}}\mathbb{E}\left[V\left(\frac{\rho_1}{\xi_{1i}}, \ldots, \frac{\rho_d}{\xi_{di}}\right)\right]\overline{F}(x). \qquad (17)$$

Lemma 4. *Under the conditions of Theorem 1, we have*

$$\mathbb{P}\left(\bigcap_{k=1}^{d}\left\{\sum_{i=1}^{N_k(T)} X_{ki}\mathrm{e}^{-r\tau_{ki}} > \rho_k x\right\}\right)$$

$$\sim\left[\int_{0-}^{T}\cdots\int_{0-}^{T} V\big(\rho_1\mathrm{e}^{rt_1},\ldots,\rho_d\mathrm{e}^{rt_d}\big)\,\lambda\left(\mathrm{d}t_1,\ldots,\mathrm{d}t_d\right)\right]\overline{F}\left(x\right).$$

Lemma 5. *Let* $\{Y_i; i \geq 1\}$ *be a sequence of i.i.d. nonnegative random variables with common distribution function* F *such that* $\overline{F} \in \mathcal{R}_{-\alpha}$. *Let* $\{\tau_i; i \geq 1\}$ *be the sequence of claim-arrival times of a renewal process. Then, for any* $r > 0$, *we have*

$$\lim_{N\to\infty} \limsup_{x\to\infty} \frac{\mathbb{P}\left(\sum_{i=N}^{\infty} Y_i \mathrm{e}^{-r\tau_i} > x\right)}{\overline{F}(x)} = 0.$$

Lemma 6. *Under the conditions of Theorem 2, it holds uniformly for $n \geq 1$ that*

$$\mathbb{P}\left(\bigcap_{k=1}^{d}\left\{\sum_{i=1}^{n} X_{ki}\mathrm{e}^{-r\tau_{ki}} > \rho_k x\right\}\right) \sim \sum_{i=1}^{n} \mathbb{E}\left[V\left(\rho_1 \mathrm{e}^{r\tau_{1i}}, \ldots, \rho_d \mathrm{e}^{r\tau_{di}}\right)\right] \overline{F}(x). \quad (18)$$

5. References.

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Thank you!