Asymptotic Ruin Probabilities for a Multidimensional Renewal Risk Model with Multivariate Regularly Varying Claims

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1. Introduction.

Consider an insurance company which simultaneously operates $d$ kinds of businesses. Its surplus process can be described by the following multidimensional risk model:

$$
\begin{pmatrix}
U_1(t) \\
\vdots \\
U_d(t)
\end{pmatrix} = 
\begin{pmatrix}
\rho_1 xe^{rt} \\
\vdots \\
\rho_d xe^{rt}
\end{pmatrix} + 
\begin{pmatrix}
c_1 \int_0^t e^{r(t-s)} ds \\
\vdots \\
c_d \int_0^t e^{r(t-s)} ds
\end{pmatrix} - 
\begin{pmatrix}
\sum_{i=1}^{N_1(t)} X_{1i} e^{r(t-\tau_{1i})} \\
\vdots \\
\sum_{i=1}^{N_d(t)} X_{di} e^{r(t-\tau_{di})}
\end{pmatrix},
$$

where $\{(U_1(t), \ldots, U_d(t)); t \geq 0\}$ denotes the multidimensional surplus process, $r \geq 0$ the constant force of interest, $(\rho_1 x, \ldots, \rho_d x)$ the vector of initial surpluses assigned to different businesses with positive $\rho_1$, $\ldots$, $\rho_d$ such that $\sum_{k=1}^d \rho_k = 1$, $(c_1, \ldots, c_d)$ the vector of constant premium rates, $\{(X_{1i}, \ldots, X_{di}); i \geq 1\}$ the sequence of claim-size vectors, and $\tau_{k1}, \tau_{k2}, \ldots$ the claim-arrival times of the $k$th business with the corresponding claim-number process $\{N_k(t); t \geq 0\}$ for $k = 1, \ldots, d$. 
Define the finite-time and infinite-time ruin probabilities corresponding to risk model (1) as

\[ \psi(x; T) = \mathbb{P} \left( T_{\text{max}} \leq T \mid (U_1(0), \ldots, U_d(0)) = (\rho_1 x, \ldots, \rho_d x) \right), \]

and

\[ \psi(x) = \mathbb{P} \left( T_{\text{max}} < \infty \mid (U_1(0), \ldots, U_d(0)) = (\rho_1 x, \ldots, \rho_d x) \right), \]

where

\[ T_{\text{max}} = \inf \{ t > 0 : \max \{ U_1(t), \ldots, U_d(t) \} < 0 \} \]

denotes the ruin time with \( \inf \emptyset = \infty \) by convention.
The present paper is devoted to extend the existing works from three main aspects. First, we drop the restriction that all businesses have a totally identical claim-number process, and instead introduce a general dependence structure into the different claim-number processes. Second, we synthetically model the dependence and heavy-tailed nature of claim sizes by a unified framework of multivariate regular variation. Last but not the least, we obtain asymptotic expansions for both finite-time and infinite-time ruin probabilities.
2. Preliminaries.

A distribution function $F = 1 - \overline{F}$ on $[0, \infty)$ is said to belong to the class of regular variation if $\overline{F}(x) > 0$ for all $x \geq 0$ and the relation

$$\lim_{x \to \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\alpha}, \quad y > 0, \quad (2)$$

holds for some $0 < \alpha < \infty$. We signify the regularity property in (2) as $\overline{F} \in \mathcal{R}_{-\alpha}$. 
For a distribution function $F$ with $F \in \mathcal{R}_{-\alpha}$ for some $0 < \alpha < \infty$, we know that, for any $0 < p_1 < \alpha < p_2 < \infty$ and $C > 1$, there is some $D > 0$ such that the inequalities

$$\frac{1}{C} \min \{y^{-p_1}, y^{-p_2}\} \leq \frac{F(xy)}{F(x)} \leq C \max \{y^{-p_1}, y^{-p_2}\}$$

(3)

hold whenever $x > D < xy$. We can derive from (3) that if $F \in \mathcal{R}_{-\alpha}$ then, for any $p > \alpha$,

$$x^{-p} = o\left(\frac{F(x)}{F(x)}\right), \quad x \to \infty.$$ (4)

It is known that a distribution function $F$ with a regularly varying tail belongs to the long-tailed distribution class $\mathcal{L}$ characterized by $F(x) > 0$ for all $x \geq 0$ and the relation

$$\lim_{x \to \infty} \frac{F(x + y)}{F(x)} = 1, \quad y \in (-\infty, \infty).$$
Consider the multidimensional risk model (1). Throughout, we assume that 
\( \{(N_1(t), \ldots, N_d(t)); t \geq 0\} \) and \( \{(X_{1i}, \ldots, X_{di}); i \geq 1\} \) are mutually independent.
For \( k = 1, \ldots, d \), denote by \( \theta_{k1} = \tau_{k1} \) and \( \theta_{ki} = \tau_{ki} - \tau_{ki-1} \) for \( i = 2, 3, \ldots \) the inter-arrival times of claims from the \( k \)th business. We introduce a general dependence structure into the different claim-number processes through the following assumption:

**Assumption 2.1.** \( \{(\theta_1, \ldots, \theta_d), (\theta_{1i}, \ldots, \theta_{di}); i \geq 1\} \) is a sequence of independent and identically distributed (i.i.d.) nonnegative random vectors, but the \( d \) components of each vector can be arbitrarily dependent.
Clearly, under Assumption 2.1 all of \( \{N_1(t); t \geq 0\} , \ldots , \{N_d(t); t \geq 0\} \) are traditional renewal processes, and they inherit dependences from that among \( \theta_1 , \ldots , \theta_d \). Further, for \( (t_1 , \ldots , t_d) \in \mathbb{R}_+^d \), we write

\[
N (t_1 , \ldots , t_d) = \max \{i : \tau_{1i} \leq t_1 , \ldots , \tau_{di} \leq t_d\} = \min \{N_k(t_k)\}_{1 \leq k \leq d}
\]

(5)

and

\[
\lambda (t_1 , \ldots , t_d) = \mathbb{E} (N (t_1 , \ldots , t_d)) = \sum_{i=1}^{\infty} \mathbb{P} (\tau_{1i} \leq t_1 , \ldots , \tau_{di} \leq t_d),
\]

(6)

which are called in the literature the \( d \)-dimensional renewal process and the corresponding renewal function respectively.
Before modeling the dependence structure among the claim sizes, we need to introduce the concept of multivariate regular variation (MRV) first. A random vector \((Z_1, \ldots, Z_d)\) taking values in \([0, \infty]^d \setminus \{\mathbf{0}\}\) is said to follow a distribution with a multivariate regularly varying tail if there exist some \(0 < \alpha < \infty\), some distribution function \(F\) with \(\overline{F} \in \mathcal{R}_{-\alpha}\), and some Radon measure \(\nu\) on \([0, \infty]^d \setminus \{\mathbf{0}\}\) satisfying 
\[
\nu \left( [0, \infty]^d \setminus \{\mathbf{0}\} \right) > 0
\]
such that the following vague convergence holds as \(x \to \infty\):

\[
\frac{1}{\overline{F}(x)} \mathbb{P} \left( \left( \frac{Z_1, \ldots, Z_d}{x} \right) \in \cdot \right) \xrightarrow{v} \nu (\cdot) \quad \text{on} \quad [0, \infty]^d \setminus \{\mathbf{0}\}.
\]

In this case, we write \((Z_1, \ldots, Z_d) \in \text{MRV} (\alpha, F, \nu)\).

Now we model the dependences among the claim sizes from different businesses via the framework of MRV. Concretely speaking, we assume:

**Assumption 2.2.** \(\{(X_1, \ldots, X_d), (X_{1i}, \ldots, X_{di}); i \geq 1\}\) is a sequence of i.i.d. nonnegative random vectors with \((X_1, \ldots, X_d) \in \text{MRV} (\alpha, F, \nu)\) such that 
\(\nu ((1, \infty]) > 0\).
We can derive from Assumption 2.2 that

\[ \nu(sK) = s^{-\alpha} \nu(K) \text{ for } s \in (0, \infty) \text{ and Borel set } K \subset [0, \infty]^d \setminus \{0\}, \quad (7) \]

\[ \lim_{x \to \infty} \frac{1}{F(x)} \mathbb{P} \left( \bigcap_{k=1}^{d} \{ X_k > x \} \right) = \nu ((1, \infty]) > 0, \quad (8) \]

and

\[ \lim_{x \to \infty} \frac{\mathbb{P}(X_k > x)}{F(x)} = \nu ((1_k, \infty]) =: a_k > 0, \quad k = 1, \ldots, d, \quad (9) \]

where \( 1_k \) is the vector with the \( k \)th element being 1 and the other elements being 0.
Relation (9) indicates that the tails of $X_1, \ldots, X_d$ are regularly varying and mutually comparable. This fact, combined with (8), implies that $X_1, \ldots, X_d$ are pairwise asymptotically dependent. Additionally, we know from (7) and (8) that

$$\lim_{x \to \infty} \frac{1}{F(x)} \mathbb{P}\left( \bigcap_{k=1}^{d} \{X_k > b_k x\} \right) = \nu \left( (b_1, \infty] \times \cdots \times (b_d, \infty]\right) > 0$$

(10)

holds for any $(b_1, \ldots, b_d) \in [0, \infty]^d \setminus \{0\}$.

In what follows, for any $(b_1, \ldots, b_d) \in [0, \infty]^d \setminus \{0\}$, we will write

$$\nu \left( (b_1, \infty] \times \cdots \times (b_d, \infty]\right) =: V (b_1, \ldots, b_d).$$

(11)
3. Main Results.

Theorem 1. Consider risk model (1). Let Assumptions 2.1 and 2.2 hold. Then, for every $T$ such that $\lambda(T,\ldots,T) > 0$, we have

$$
\psi(x,T) \sim \left[ \int_{0}^{T} \cdots \int_{0}^{T} V(\rho_{1}e^{rt_1},\ldots,\rho_{d}e^{rt_d}) \lambda(dt_1,\ldots,dt_d) \right] F(x). \quad (12)
$$
Next, we focus on the infinite-time ruin probability. To this end, we naturally require \( r > 0 \) in risk model (1) for the convergence of quantities under consideration.

**Theorem 2.** In addition to the other conditions of Theorem 1, if \( r > 0 \) then (12) holds also for \( T = \infty \), i.e., we have

\[
\psi(x) \sim \left[ \int_{0-}^{\infty} \cdots \int_{0-}^{\infty} V(\rho_1 e^{rt_1}, \ldots, \rho_d e^{rt_d}) \lambda(dt_1, \ldots, dt_d) \right] \overline{F}(x). \tag{13}
\]
Particularly, if the businesses share a common claim-number process, i.e.,
\( N_1(t) \equiv \cdots \equiv N_d(t) \equiv N(t) \), then we can immediately obtain more elegant and transparent forms for (12) and (13) by applying (7).

**Corollary 1.** In addition to the other conditions of Theorem 1, if \( N_1(t) \equiv \cdots \equiv N_d(t) \equiv N(t) \) with \( \lambda(t) = \mathbb{E}(N(t)) \) then

\[
\psi(x, T) \sim \left( \int_{0}^{T} e^{-\alpha rt} \lambda(dt) \right) V(\rho_1, \ldots, \rho_d) \overline{F}(x). \tag{14}
\]

Further if \( r > 0 \) then

\[
\psi(x) \sim \frac{\mathbb{E}(e^{-\alpha r \theta})}{1 - \mathbb{E}(e^{-\alpha r \theta})} V(\rho_1, \ldots, \rho_d) \overline{F}(x), \tag{15}
\]

where \( \theta \) is the generic random variable of the inter-arrival times of \( \{N(t); t \geq 0\} \).
Formulas (12)–(15) reveal that the ruin probabilities of risk model (1) with our dependence structures assume a form of some constant times $F(x)$. Although the constants before $F(x)$ are involved in general, the formulas enable us to easily conduct numerical estimates for the ruin probabilities;
4. Lemmas.

Lemma 2. Let random vector \((Z_1, \ldots, Z_d) \in \text{MRV} (\alpha, F, \nu)\), and let \((\xi_1, \ldots, \xi_d)\) be a nonnegative random vector with arbitrarily dependent components satisfying \(\mathbb{E} (\xi_k^p) < \infty\) for some \(p > \alpha\) and all \(1 \leq k \leq d\). Assume that \((\xi_1, \ldots, \xi_d)\) and \((Z_1, \ldots, Z_d)\) are independent. Then, for every Borel set \(K \subset [0, \infty]^d \setminus \{0\}\), we have

\[
\lim_{x \to \infty} \frac{1}{F(x)} \mathbb{P} \left( \left( \frac{\xi_1 Z_1, \ldots, \xi_d Z_d}{x} \right) \in K \right) = \mathbb{E} \left[ \nu (\xi^{-1} K) \right],
\]

(16)

where \(\xi^{-1} K = \{(b_1, \ldots, b_d) | (\xi_1 b_1, \ldots, \xi_d b_d) \in K\}\).
Lemma 3. Let the conditions of Theorem 1 hold, and let \( \{\xi_{ki}; 1 \leq k \leq d, i \geq 1\} \) be a sequence of nonnegative and arbitrarily dependent random variables satisfying \( E(\xi_{ki}^p) < \infty \) for some \( p > \alpha \) and all \( 1 \leq k \leq d, i \geq 1 \). Assume that \( \{\xi_{ki}; 1 \leq k \leq d, i \geq 1\} \) and \( \{(X_{1i}, \ldots, X_{di}); i \geq 1\} \) are mutually independent. Then, for any \( n_1 \geq 1, \ldots, n_d \geq 1 \) with \( \hat{n} = \min_{1 \leq k \leq d} \{n_k\} \), we have

\[
\mathbb{P}\left(\bigcap_{k=1}^{d} \left\{ \sum_{i=1}^{n_k} \xi_{ki}X_{ki} > \rho_k x \right\}\right) \sim \sum_{i=1}^{\hat{n}} \mathbb{E}\left[ V\left(\frac{\rho_1}{\xi_{1i}}, \ldots, \frac{\rho_d}{\xi_{di}}\right)\right] F(x). \quad (17)
\]
Lemma 4. Under the conditions of Theorem 1, we have

\[
\mathbb{P} \left( \bigcap_{k=1}^{d} \left\{ \sum_{i=1}^{N_k(T)} X_{ki} e^{-r\tau_{ki}} > \rho_k x \right\} \right)
\]

\[
\sim \left[ \int_{0^-}^{T} \cdots \int_{0^-}^{T} V(\rho_1 e^{rt_1}, \ldots, \rho_d e^{rt_d}) \lambda(dt_1, \ldots, dt_d) \right] F(x).
\]
Lemma 5. Let \( \{Y_i; i \geq 1\} \) be a sequence of i.i.d. nonnegative random variables with common distribution function \( F \) such that \( \overline{F} \in \mathcal{R}_{-\alpha} \). Let \( \{\tau_i; i \geq 1\} \) be the sequence of claim-arrival times of a renewal process. Then, for any \( r > 0 \), we have
\[
\lim_{N \to \infty} \limsup_{x \to \infty} \frac{\mathbb{P} \left( \sum_{i=N}^{\infty} Y_i e^{-r \tau_i} > x \right)}{F(x)} = 0.
\]
Lemma 6. Under the conditions of Theorem 2, it holds uniformly for $n \geq 1$ that

\[ P \left( \bigcap_{k=1}^{d} \left\{ \sum_{i=1}^{n} X_{ki} e^{-r\tau ki} > \rho_k x \right\} \right) \sim \sum_{i=1}^{n} \mathbb{E} \left[ V(\rho_1 e^{r\tau_1 i}, \ldots, \rho_d e^{r\tau di}) \right] F(x). \] (18)
5. References.

References


Thank you!