BIVARIATE HIGHER-DEGREE INCREASING CONVEX ORDERS AND RELATED DEPENDENCE CONCEPTS, WITH APPLICATIONS TO RISK AND INSURANCE

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Univariate stochastic dominance rules, D, Lefèvre and Shaked (1998)

Bivariate stochastic dominance rules, D, Lefèvre and Mesfioui (1999)

Expectation dependence and related concepts

Application 1: Asset allocation by prudent investors, D and Eeckhoudt (2016)

Application 2: Risk aversion with two risks, D and Mesfioui (2016)

Application 3: Model selection based on convex order, D and Trufin (2016)

Some references

Outline

Univariate stochastic dominance rules, D, Lefèvre and Shaked (1998)

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Increasing convex/concave orders of various degrees

- Consider two risks X and Y, i.e. non-negative random variables representing financial losses.
- D, Lefèvre and Shaked (1998):

$$X \preceq_{s-\text{icx}} Y \Leftrightarrow \begin{cases} \mathrm{E}[(X-t)_+^{s-1}] \leq \mathrm{E}[(Y-t)_+^{s-1}] \text{ for all } t \geq 0 \\ \mathrm{E}[X^k] \leq \mathrm{E}[Y^k] \text{ for all } k = 1, \ldots, s-1. \end{cases}$$

$$\Leftrightarrow \mathrm{E}[g(X)] \leq \mathrm{E}[g(Y)] \text{ for all } g \text{ such that } g^{(k)} \geq 0 \text{ for } k = 1, \ldots, s.$$

- Also,

$$X \preceq_{s-\text{icx}} Y \Leftrightarrow \mathrm{E}[u(w-X)] \geq \mathrm{E}[u(w-Y)] \text{ for all initial wealth } w$$

and for every utility function u such that

$$(-1)^{k+1} u^{(k)} \geq 0 \text{ for } k = 1, \ldots, s.$$

- Define the class $\mathcal{U}_{s-\text{icv}}$, $s = 1, 2, \ldots$, of the s-increasing concave utility functions as the class containing all the functions u such that

$$(-1)^{k+1} u^{(k)} \geq 0 \text{ for } k = 1, \ldots, s.$$

- Letting s tend to $+\infty$ gives completely monotone utilities expressing mixed risk aversion, as studied in Caballe and Pomansky (1996).
- The common preferences of all the decision makers with s-increasing concave utility functions generate $\preceq_{s-\text{icv}}$:

$$X \preceq_{s-\text{icv}} Y \quad \Leftrightarrow \quad \mathrm{E}[u(X)] \leq \mathrm{E}[u(Y)] \text{ for all } u \text{ in } \mathcal{U}_{s-\text{icv}},$$

provided the expectations exist.

Particular cases: $s \in \{1, 2\}$

$$
\begin{aligned}
X \preceq_{1-\text{icx}} Y \quad &\Leftrightarrow \quad X \preceq_{1-\text{icv}} Y \\
&\Leftrightarrow \quad \mathrm{P}[X \leq t] \geq \mathrm{P}[Y \leq t] \text{ for all } t \\
&\Leftrightarrow \quad \mathrm{P}[X > t] \leq \mathrm{P}[Y > t] \text{ for all } t \\
&\Leftrightarrow \quad \mathrm{E}[g(X)] \leq \mathrm{E}[g(Y)] \text{ for all } \nearrow g
\end{aligned}
$$

$$
\begin{aligned}
X \preceq_{2-\text{icx}} Y \quad &\Leftrightarrow \quad \mathrm{E}[(X - t)_+] \leq \mathrm{E}[(Y - t)_+] \text{ for all } t \\
&\Leftrightarrow \quad \mathrm{E}[g(X)] \leq \mathrm{E}[g(Y)] \text{ for all } \nearrow \text{ convex } g
\end{aligned}
$$

$$
\begin{aligned}
X \preceq_{2-\text{icv}} Y \quad &\Leftrightarrow \quad \mathrm{E}[(t - X)_+] \geq \mathrm{E}[(t - Y)_+] \quad \text{for all } t \\
&\Leftrightarrow \quad \mathrm{E}[u(X)] \leq \mathrm{E}[u(Y)] \text{ for all } \nearrow \text{ concave } u.
\end{aligned}
$$

Risk apportionment of degree s

- According to Eeckhoudt and Schlesinger (2006), preferences are said to satisfy risk apportionment of degree s if

$$(-1)^{s+1}u^{(s)} \geq 0 \Leftrightarrow \text{preference for "pain disaggregation"}.$$

- This notion extends
  non-satiation $(s = 1)$ i.e. more money is better.
  risk aversion $(s = 2)$ i.e. the preference for $\mathrm{E}[X]$ over X, whatever X.
  prudence $(s = 3)$ i.e. building up precautionary savings in order to better face future income risk.
  as well as temperance $(s = 4)$ and edginess $(s = 5)$ to any degree s.
- See D, Lefèvre and Scarsini (2001) for early illustrations in terms of lotteries.

Risk apportionment $\Leftrightarrow$ "combining good with bad", Eeckhoudt, Schlesinger and Tsetlin (2009)

- Consider independent random variables X_1, X_2, Y_1, and Y_2, such that
$$X_1 \preceq_{s_1-\text{icv}} Y_1 \text{ and } X_2 \preceq_{s_2-\text{icv}} Y_2.$$
- Provided $u \in \mathcal{U}_{(s_1+s_2)-\text{icv}}$, the lottery
$$\mathcal{L} = \begin{cases} X_1 + Y_2 \text{ with probability } \frac{1}{2} \\ Y_1 + X_2 \text{ with probability } \frac{1}{2} \end{cases}$$
$$\text{is preferred to } \mathcal{M} = \begin{cases} X_1 + X_2 \text{ with probability } \frac{1}{2} \\ Y_1 + Y_2 \text{ with probability } \frac{1}{2} \end{cases}.$$

Risk apportionment ⇔ "(positive) correlation aversion" decreasing in wealth, D and Rey (2010)

- Consider I_1 and I_2 such that

$$\mathrm{P}[I_i = 0] = 1 - \mathrm{P}[I_i = 1] = p_i, \ \ i = 1, 2.$$

- Define the joint distribution of (I_1, I_2) as

$$\begin{aligned}
\mathrm{P}[I_1 = 0, I_2 = 0] &= p_1 p_2 + \rho \\
\mathrm{P}[I_1 = 1, I_2 = 0] &= (1 - p_1)p_2 - \rho \\
\mathrm{P}[I_1 = 0, I_2 = 1] &= p_1(1 - p_2) - \rho \\
\mathrm{P}[I_1 = 1, I_2 = 1] &= (1 - p_1)(1 - p_2) + \rho
\end{aligned}$$

  for some suitable correlation parameter ρ.

- Consider independent random variables X_1, X_2, Y_1, and Y_2, independent from (I_1, I_2), and such that $X_1 \preceq_{s_1-\text{icv}} Y_1$ and $X_2 \preceq_{s_2-\text{icv}} Y_2$.

- Consider $w + (1 - I_1)X_1 + I_1 Y_1 + (1 - I_2)X_2 + I_2 Y_2$

$$= \begin{cases} w + X_1 + X_2 \text{ with probability } p_1 p_2 + \rho, \\ w + X_1 + Y_2 \text{ with probability } p_1(1 - p_2) - \rho, \\ w + Y_1 + X_2 \text{ with probability } (1 - p_1)p_2 - \rho, \\ w + Y_1 + Y_2 \text{ with probability } (1 - p_1)(1 - p_2) + \rho. \end{cases}$$

- D, Eeckhoudt and Rey (2010): if $u \in \mathcal{U}_{(s_1+s_2)-\text{icv}}$ then the decision maker dislikes an increase in the correlation parameter ρ, i.e.

$$w+(1-I_1)X_1+I_1 Y_1+(1-I_2)X_2+I_2 Y_2 \searrow \text{ in } \rho \text{ wrt } \preceq_{(s_1+s_2)-\text{icv}} .$$

- D and Rey (2010) proved that provided $u \in \mathcal{U}_{(s_1+s_2+1)-\text{icv}}$,

$$\frac{\partial^2}{\partial w \partial \rho} \mathrm{E}[u(w + (1 - I_1)X_1 + I_1 Y_1 + (1 - I_2)X_2 + I_2 Y_2)] \geq 0.$$

  - This means that such decision makers become less sensitive to an increase in correlation when they get richer.

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Bivariate extension by D, Lefèvre and Mesfioui (1999)

- Consider pairs of random variables (X, Y) valued in $[a_1, b_1] \times [a_2, b_2]$.
- Let $\mathcal{U}_{(s_1,s_2)-\text{icv}}$ be the class of the regular (s_1, s_2)-increasing concave functions, that is, those functions u such that

$$(-1)^{k_1+k_2+1} u^{(k_1,k_2)} \geq 0$$

  for all $k_1 = 0, \ldots, s_1$, $k_2 = 0, \ldots, s_2$, such that $k_1 + k_2 \geq 1$.
- Then,

  $(X_1, Y_1) \preceq_{(s_1,s_2)-\text{icv}} (X_2, Y_2)$

$$\Leftrightarrow \mathrm{E}[u(X_1, Y_1)] \leq \mathrm{E}[u(X_2, Y_2)] \text{ for all } u \in \mathcal{U}_{(s_1,s_2)-\text{icv}},$$

  provided the expectations exist.

Meaning of utilities in $\mathcal{U}_{(s_1,s_2)-\text{icv}}$

- Eeckhoudt, Rey and Schlesinger (2007) proved equivalence between the signs of the partial derivatives $u^{(k_1,k_2)}$ and individual preferences within a particular class of simple lotteries.
- This lead to the concept of cross risk apportionment.
- Specifically, an individual is said to be
  correlation averse $\Leftrightarrow u^{(1,1)} \leq 0$,
  cross-prudent in the second outcome $\Leftrightarrow u^{(2,1)} \geq 0$,
  cross-prudent in the first outcome $\Leftrightarrow u^{(1,2)} \geq 0$, and
  cross-temperant $\Leftrightarrow u^{(2,2)} \leq 0$.
- The meaning of utilities in $\mathcal{U}_{(s_1,s_2)-\text{icv}}$ for higher values of s_1 and s_2 has been made clear in D and Eeckhoudt (2010) and D and Rey (2013).

Cross risk apportionment $\Leftrightarrow$ "combining good with bad", Jokung (2011)

- Consider independent random variables X_1, X_2, Y_1, and Y_2, such that
$$X_1 \preceq_{s_1-\text{icv}} Y_1 \text{ and } X_2 \preceq_{s_2-\text{icv}} Y_2.$$
- Provided $u \in \mathcal{U}_{(s_1,s_2)-\text{icv}}$, the lottery
$$\mathcal{L} = \begin{cases} (X_1, Y_2) \text{ with probability } \frac{1}{2} \\ \\ (Y_1, X_2) \text{ with probability } \frac{1}{2} \end{cases}$$
$$\text{is preferred to } \mathcal{M} = \begin{cases} (X_1, X_2) \text{ with probability } \frac{1}{2} \\ \\ (Y_1, Y_2) \text{ with probability } \frac{1}{2} \end{cases}.$$

Particular cases: $s_i \in \{1, 2\}$

$$(X_1, Y_1) \preceq_{(1,1)-\mathrm{icv}} (X_2, Y_2)$$

$$\Leftrightarrow \mathrm{P}[X_1 \le t_1, Y_1 \le t_2] \ge \mathrm{P}[X_2 \le t_1, Y_2 \le t_2] \text{ for all } t_1 \text{ and } t_2.$$

$$(X_1, Y_1) \preceq_{(2,1)-\mathrm{icv}} (X_2, Y_2)$$

$$\Leftrightarrow \begin{cases} Y_1 \preceq_{1-\mathrm{icv}} Y_2 \\ \mathrm{E}\Big[(t_1 - X_1)_+ \mathrm{I}[Y_1 \le t_2]\Big] \ge \mathrm{E}\Big[(t_1 - X_2)_+ \mathrm{I}[Y_2 \le t_2]\Big] \\ \quad \text{for all } t_1 \text{ and } t_2 \ \ (\Rightarrow X_1 \preceq_{2-\mathrm{icv}} X_2). \end{cases}$$

$$(X_1, Y_1) \preceq_{(2,2)-\mathrm{icv}} (X_2, Y_2)$$

$$\Leftrightarrow \begin{cases} X_1 \preceq_{2-\mathrm{icv}} X_2 \\ Y_1 \preceq_{2-\mathrm{icv}} Y_2 \\ \mathrm{E}\Big[(t_1 - X_1)_+ (t_2 - Y_1)_+\Big] \ge \mathrm{E}\Big[(t_1 - X_2)_+ (t_2 - Y_2)_+\Big] \\ \quad \text{for all } t_1 \text{ and } t_2. \end{cases}$$

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Expectation dependence

- According to Wright (1987), the random variable X is negatively expectation dependent on Y if

$$E[X] \leq E[X|Y \leq t] \text{ for all } t.$$

$$\Leftrightarrow E[X|Y > t] \leq E[X] \text{ for all } t.$$

- The equivalent definition corresponds to the negative expectation quadrant dependence introduced by Kowalczyk and Pleszczynska (1977).
- Positive expectation dependence is defined by reversing the sign of the inequality, i.e.

$$E[X] \geq E[X|Y \leq t] \text{ for all } t$$

$$\Leftrightarrow E[X] \leq E[X|Y > t] \text{ for all } t.$$

Key properties of expectation dependence

- X is negatively expectation dependent on Y, i.e.,

$$\mathrm{E}[X|Y>t]-\mathrm{E}[X]\leq 0 \text{ for all } t$$

if, and only if,

$$\mathrm{E}\big[X\mathrm{I}[Y>t]\big]-\mathrm{E}[X]\mathrm{E}\big[\mathrm{I}[Y>t]\big]\leq 0 \text{ for all } t$$

$$\Leftrightarrow \mathrm{C}\big[X,\mathrm{I}[Y>t]\big]\leq 0 \text{ for all } t$$

$$\Leftrightarrow \mathrm{C}[X,g(Y)]\leq 0 \text{ for all } \nearrow g.$$

- Useful identity:

$$\mathrm{C}[X,g(Y)]=\int_{a_2}^{b_2}\Big(\mathrm{E}[X]-\mathrm{E}[X|Y\leq t]\Big)\mathrm{P}[Y\leq t]g^{(1)}(t)dt.$$

Example, Egozcue et al. (2011, 2013)

- Consider unit uniform X and Y with joint distribution function C.
- Expectation dependence controls the sign of

$$\Delta(y) = \int_0^1 \Big(C(x,y) - xy \Big) dx.$$

- Consider

$$C(x,y) = (1-\alpha)\max\{0, x+y-1\} + \alpha \min\{x,y\}.$$

- Then,

$$\Delta(y) = y(1-y)\left(\alpha - \frac{1}{2}\right).$$

- More generally, expectation dependence can be obtained by mixing positive and negative quadrant dependent distributions.

Second-degree expectation dependence

- Define $ED_1(X|y) = \mathrm{E}[X] - \mathrm{E}[X|Y \leq y]$.
- First-degree expectation dependence controls the sign of $ED_1(X|y)$.
- Li (2011): X is positively second-degree expectation dependent on Y if

$$\begin{aligned} ED_2(X|y) &= \int_{a_2}^{y} ED_1(X|t)\mathrm{P}[Y \leq t]dt \\ &= -\mathrm{C}[X, (y - Y)_+] \geq 0 \text{ for all } y \end{aligned}$$

$$\Leftrightarrow \mathrm{C}[X, (y - Y)_+] \leq 0 \text{ for all } y.$$

- Similarly, X is negatively second-degree expectation dependent on Y if

$$\mathrm{C}[X, (y - Y)_+] \geq 0 \text{ for all } y.$$

Second-degree excess dependence

- First-degree expectation dependence can be equivalently defined by means of conditionings of the form $Y \leq y$ or $Y > y$.
- However, this is not the case for second-degree expectation dependence where these two conditions lead to different concepts.
- This is why D, Huang and Tzeng (2015) introduced a dual concept of dependence corresponding to the excess condition $Y > y$.
- Precisely, define

$$\overline{ED}_1(X|Y) = \mathrm{E}[X|Y > y] - \mathrm{E}[X].$$

- First-degree expectation dependence controls the sign of $\overline{ED}_1$.

- Now, let us integrate to get

$$\overline{ED}_2(X|y) \;=\; \int_y^{b_2} \overline{ED}_1(X|t)\mathrm{P}[Y > t]dt = \mathrm{C}[X,(Y-y)_+].$$

- Then, X is positively second-degree excess dependent on Y if

$$\overline{ED}_2(X|y) \geq 0 \text{ for all } y \Leftrightarrow \mathrm{C}\big[X,(Y-y)_+\big] \geq 0 \text{ for all } y.$$

- Property: X positively second-degree excess dependent on Y

$$\Rightarrow \mathrm{C}[X, g(Y)] \geq 0 \text{ for every } \nearrow \text{ convex } g$$

  for which the covariances exist.
- See Li (2011) and D, Huang and Tzeng (2015) for extensions to degrees 3 and higher.

Positive dependence notions

- Let $(X^{\perp}, Y^{\perp})$ be an independent version of (X, Y), i.e. $(X^{\perp}, Y^{\perp})$ is such that
  - X and $X^{\perp}$ are identically distributed,
  - Y and $Y^{\perp}$ are identically distributed, and
  - $X^{\perp}$ and $Y^{\perp}$ are mutually independent.
- Expectation dependence can be defined by comparing

$$\mathrm{E}[X|Y \leq y] - \mathrm{E}[X] \text{ with } \mathrm{E}[X^{\perp}|Y^{\perp} \leq y] - \mathrm{E}[X^{\perp}] = 0.$$

- Comparing (X, Y) to $(X^{\perp}, Y^{\perp})$ using $\preceq_{(s_1,s_2)-\mathrm{icv}}$ allows to get relevant positive dependence notions, as proposed by D, Lefèvre and Mesfioui (1999):
  - with $\preceq_{(1,1)-\mathrm{icv}}$, we recover the quadrant dependence of Lehmann (1966)
  - with $\preceq_{(2,1)-\mathrm{icv}}$, we get the second-degree quadratic dependence (SQD) proposed by Li et al. (2015, Definition 3.5).
- Clearly, $(X, Y) \preceq_{(2,1)-\mathrm{icv}} (X^{\perp}, Y^{\perp}) \Rightarrow X$ is positively expectation dependent on Y.

Positive stop-loss dependence, D, Dhaene, Goovaerts and Kaas (2005, Chapter 5)

- Idea: the stop-loss premium for X is increased when it is known that Y is large, i.e.

  $(X^{\perp}, Y^{\perp}) \preceq_{(2,1)-\mathrm{icx}} (X, Y)$

  $$\Leftrightarrow \mathrm{E}[(X-d)_+] \leq \mathrm{E}[(X-d)_+ | Y > t] \text{ for all } t \text{ and } d$$

  $$\Leftrightarrow X \preceq_{2-\mathrm{icx}} [X \text{ given } Y > t].$$

- For $d = 0$ we recover expectation dependence:

  $$\mathrm{E}[X] \leq \mathrm{E}[X | Y > t] \text{ for all } t.$$

- Here, we get

  $(X^{\perp}, Y^{\perp}) \preceq_{(2,1)-\mathrm{icx}} (X, Y)$

  $\Leftrightarrow \mathrm{C}[g_1(X), g_2(Y)] \geq 0$ for all $\nearrow$ g_1 and $g_2 \geq 0$, with g_1 convex.

Outline

Two-asset portfolio problem

- Let X_j, $j = 1, 2$, be the random return per monetary unit invested in risky asset j valued in some interval $[a, b]$ of the real line.
- Assume that the initial wealth is equal to unity and must be invested in one of these two risky assets by a non-satiated risk-averse decision maker.
- Let λ denote the fraction of the initial wealth invested in X_1.
- For an investor with utility function $u \in \mathcal{U}_{2-\text{icv}}$, the optimal λ maximizes the objective function

$$\mathcal{O}(\lambda) = \mathrm{E}[u(\overline{Y}_\lambda)] \text{ with } \overline{Y}_\lambda = \lambda X_1 + (1 - \lambda) X_2.$$

- The first-order condition is

$$\mathcal{O}'(\lambda) = 0 \Leftrightarrow \mathrm{E}[(X_1 - X_2) u'(\overline{Y}_\lambda)] = 0.$$

Minimum demand condition for risk-averse investors, Hadar and Seo (1988)

- Consider a given percentage $\pi \in [0, 1]$.
- Define the reference portfolio

$$\overline{Y}_\pi = \pi X_1 + (1 - \pi) X_2.$$

- The optimal share $\lambda^\star$ invested in the first asset is at least equal to π for every $u \in \mathcal{U}_{2-\text{icv}}$ if, and only if,

$$\mathsf{E}\big[X_1 \mathsf{I}[\overline{Y}_\pi \le z]\big] \ge \mathsf{E}\big[X_2 \mathsf{I}[\overline{Y}_\pi \le z]\big] \text{ for all } z \in [a, b]$$

$$\Leftrightarrow \mathsf{E}[X_1 | \overline{Y}_\pi \le z] \ge \mathsf{E}[X_2 | \overline{Y}_\pi \le z] \text{ for all } z \in [a, b]$$

$$\Leftrightarrow \mathsf{C}\big[X_1 - X_2, \mathsf{I}[\overline{Y}_\pi \le z]\big] \ge \mathsf{E}[X_2 - X_1] \mathsf{P}[\overline{Y}_\pi \le z] \text{ for all } z \in [a, b]$$

$$\Leftrightarrow \mathsf{E}\big[X_1 \mathsf{I}[\overline{Y}_\pi \le z]\big] \ge \mathsf{E}\big[\overline{Y}_\pi \mathsf{I}[\overline{Y}_\pi \le z]\big] \text{ for all } z \in [a, b].$$

Particular cases

- If $\mathrm{E}[X_1] = \mathrm{E}[X_2]$ then only the covariance remains and one obtains

  $$\lambda^\star \geq \pi \Leftrightarrow \mathrm{C}\big[X_1, \mathrm{I}[\overline{Y}_\pi \leq z]\big] \geq \mathrm{C}\big[X_2, \mathrm{I}[\overline{Y}_\pi \leq z]\big] \text{ for all } z \in [a, b].$$

- As $\mathrm{E}[X_1] = \mathrm{E}[X_2] = \mathrm{E}[\overline{Y}_\pi]$ also holds,

  $$\lambda^\star \geq \pi \Leftrightarrow \mathrm{C}\big[X_1, \mathrm{I}[\overline{Y}_\pi \leq z]\big] \geq \mathrm{C}\big[\overline{Y}_\pi, \mathrm{I}[\overline{Y}_\pi \leq z]\big] \text{ for all } z \in [a, b].$$

- The particular case

  $$\pi = 0 \Leftrightarrow \text{positive demand}$$

  has been considered by Wright (1987).
- With $\pi = 0$, the reference portfolio $\overline{Y}_\pi$ reduces to $\overline{Y}_0 = X_2$.

Prudent investors: Introductory example

Consider assets returns

$$X_1 = \begin{cases} 1.1 \text{ with probability } \frac{3}{4} \\ 1.3 \text{ with probability } \frac{1}{4} \end{cases} \quad \text{and } X_2 = \begin{cases} 1 \text{ with probability } \frac{1}{4} \\ 1.2 \text{ with probability } \frac{3}{4} \end{cases}$$

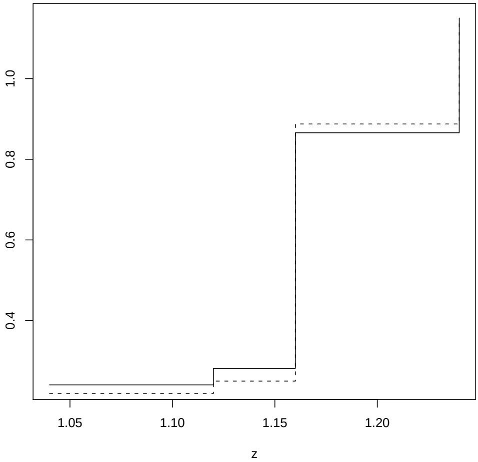
such that $\mathrm{E}[X_1] = \mathrm{E}[X_2]$, $\mathrm{V}[X_1] = \mathrm{V}[X_2]$ and $X_2 \preceq_{3-\mathrm{icv}} X_1$ with

$$\begin{aligned} \mathrm{P}[X_1 = 1.1, X_2 = 1] &= \frac{3}{16} + \rho \\ \mathrm{P}[X_1 = 1.1, X_2 = 1.2] &= \frac{9}{16} - \rho \\ \mathrm{P}[X_1 = 1.3, X_2 = 1] &= \frac{1}{16} - \rho \\ \mathrm{P}[X_1 = 1.3, X_2 = 1.2] &= \frac{3}{16} + \rho, \end{aligned}$$

for some correlation parameter $\rho \in \left[-\frac{3}{16}, \frac{1}{16}\right]$.

- Let us consider the case $\rho = \frac{1}{32}$ so that both returns are positively related.
- We take $\pi = 0.4$, meaning that the reference portfolio $\overline{Y}_{0.4}$ comprises 40% of unit wealth invested in asset 1 and we wonder whether this proportion should be increased.
- The curves $z \mapsto \mathrm{E}\big[X_1 \mathrm{I}[\overline{Y}_{0.4} \leq z]\big]$ and $z \mapsto \mathrm{E}\big[X_2 \mathrm{I}[\overline{Y}_{0.4} \leq z]\big]$ intersect (see next slide...).
- Hence, risk-averse investors do not unanimously agree to invest more that 40% of their initial wealth in asset 1.
- We could nevertheless wonder whether a subset of these decision makers would agree to do so.
- This is why we derive next the minimum demand conditions for the relevant subset of prudent investors.
- Then, we come back to this example to show that all these decision makers agree to increase the share of X_1 in their portfolio above 40%.

Graphs of $z \mapsto \mathrm{E}\big[X_1 \mathrm{I}[\overline{Y}_{0.4} \le z]\big]$ (continuous line —) and $z \mapsto \mathrm{E}\big[X_2 \mathrm{I}[\overline{Y}_{0.4} \le z]\big]$ (broken line - - -)



Minimum demand condition for risk-averse, prudent investors, D and Eeckhoudt (2016)

- Consider a given percentage $\pi \in [0,1]$.
- The optimal share $\lambda^\star$ invested in the first asset is at least equal to π for every $u \in \mathcal{U}_{3-\text{icv}}$ if, and only if,

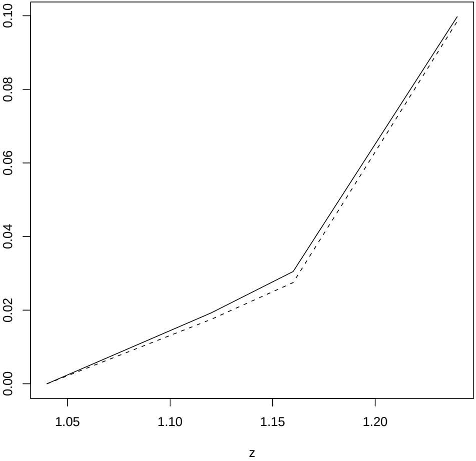
$$\mathrm{E}[X_1] \geq \mathrm{E}[X_2]$$

and one of the following equivalent conditions is fulfilled:

$$\begin{aligned} & \mathrm{E}\left[X_1(z-\overline{Y}_\pi)_+\right] \geq \mathrm{E}\left[X_2(z-\overline{Y}_\pi)_+\right] \\ \Leftrightarrow\quad & \mathrm{C}\left[X_1-X_2,(z-\overline{Y}_\pi)_+\right] \geq \mathrm{E}[X_2-X_1]\mathrm{E}[(z-\overline{Y}_\pi)_+] \\ \Leftrightarrow\quad & \mathrm{E}\left[X_1(z-\overline{Y}_\pi)_+\right] \geq \mathrm{E}\left[\overline{Y}_\pi(z-\overline{Y}_\pi)_+\right] \\ \Leftrightarrow\quad & \mathrm{C}\left[X_1-\overline{Y}_\pi,(z-\overline{Y}_\pi)_+\right] \geq \mathrm{E}[\overline{Y}_\pi-X_1]\mathrm{E}[(z-\overline{Y}_\pi)_+] \end{aligned}$$

for all $z \in [a,b]$.

Graphs of $z \mapsto \mathrm{E}\left[X_1(z - \overline{Y}_{0.4})_+\right]$ (continuous line —) and $z \mapsto \mathrm{E}\left[X_2(z - \overline{Y}_{0.4})_+\right]$ (broken line - - -)



Particular cases

- If $\mathrm{E}[X_1] = \mathrm{E}[X_2]$ then $\lambda^\star \geq \pi$ for any $u \in \mathcal{U}_{3-\text{icv}}$ when
$$\mathrm{C}\big[X_1, (z - \overline{Y}_\pi)_+\big] \geq \mathrm{C}\big[X_2, (z - \overline{Y}_\pi)_+\big] \text{ for all } z \in [a, b].$$
- As $\mathrm{E}[X_1] = \mathrm{E}[X_2] = \mathrm{E}[\overline{Y}_\pi]$ holds, $\lambda^\star \geq \pi$ for any $u \in \mathcal{U}_{3-\text{icv}}$ if
$$\mathrm{C}\big[X_1, (z - \overline{Y}_\pi)_+\big] \geq \mathrm{C}\big[\overline{Y}_\pi, (z - \overline{Y}_\pi)_+\big] \text{ for all } z \in [a, b].$$
- This condition is stated in terms of the covariance between X_1 and the put option written on the portfolio $\overline{Y}_\pi$ compared to the covariance of $\overline{Y}_\pi$ with the same put option.
- The particular case $\pi = 0$ (positive demand) has been considered by D, Huang and Tzeng (2015) who extended the analysis conducted in Wright (1987) to higher-order risk attitudes.
- Extensions to the case $u \in \mathcal{U}_{s-\text{icv}}$ with $s \geq 4$ can be found in D and Eeckhoudt (2016).

Outline

Risk aversion in presence of a background risk

- Consider a correlation-averse decision maker.
- If X and Y are negatively correlated so that X serves as a hedge against Y, then we may have

$$\mathrm{E}[u(X, Y)] \geq \mathrm{E}[u(\mathrm{E}[X], Y)].$$

- This means that the decision maker prefers to keep X as it compensates for Y.
- We need some positive dependence between X and Y to ensure that

$$\mathrm{E}[u(X, Y)] \leq \mathrm{E}[u(\mathrm{E}[X], Y)]$$

  holds true for every "reasonable" utility function u.
- Similar problems have been considered with $u \in \mathcal{U}_{(s_1,s_2)-\mathrm{icv}}$, by D, Lefèvre and Mesfioui (1999) under quadrant dependence (take $s_1 = 2$ and $s_2 = 1$ here).

- As well as with $u \in (\mathcal{U}_{(1,1)-\text{icv}} \cap \mathcal{U}_{(2,0)-\text{icv}})$ by
  Finkelshtain, Kella and Scarsini (1999) under regression dependence i.e. $y \mapsto \mathsf{E}[X|Y=y] \nearrow$
  Li et al. (2016) under expectation dependence.
- Precisely, Li et al. (2016) established that if X is positively expectation dependent on Y then

  $$\mathsf{E}[u(X,Y)] \leq \mathsf{E}[u(\mathsf{E}[X],Y)]$$

  holds true for every $u \in \left(\mathcal{U}_{(1,1)-\text{icv}} \cap \mathcal{U}_{(2,0)-\text{icv}}\right)$.
- Hence, X is positively expectation dependent on Y

  $$\Rightarrow (X,Y) \preceq_{(2,1)-\text{icv}} (\mathsf{E}[X],Y) \text{ holds true.}$$

- Such conditions have been imposed by D, Eeckhoudt and Menegatti (2011) in the study of optimal choices under correlated background risk.

Bivariate Rothschild-Stiglitz increase in risk

- Guo et al. (2016, Proposition 3.2) investigate sufficient conditions ensuring that

$$\mathrm{E}[u(X_1, Y)] \leq \mathrm{E}[u(X_2, Y)] \text{ for all } u \in \mathcal{U}_{(2,1)-\mathrm{icv}}$$

  provided $\mathrm{E}[X_1] = \mathrm{E}[X_2]$.
- This extends Li et al. (2016), considering $X_2 = \mathrm{E}[X_1]$.
- They establish that this inequality is valid if

$$\mathrm{E}[X_1] - \mathrm{E}[X_1 | X_2 \leq x_2, Y \leq y] \geq \mathrm{E}[X_2] - \mathrm{E}[X_2 | X_2 \leq x_2, Y \leq y]$$

  for all x_2 and y.
- The necessary and sufficient condition is

  $(X_1, Y) \preceq_{(2,1)-\mathrm{icv}} (X_2, Y)$

$$\Leftrightarrow \mathrm{E}\Big[(x_2 - X_1)_+ \mathrm{I}[Y \leq y]\Big] \geq \mathrm{E}\Big[(x_2 - X_2)_+ \mathrm{I}[Y \leq y]\Big] \text{ for all } y \text{ and } x_2.$$

Outline

Informative scores

- Assume that an analyst wants to predict a response Y by means of a set of covariates $X_1, \ldots, X_p$.
- These covariates are combined to form a score S.
- We assume that the distribution function F_S of S is continuous and strictly increasing.
- The score S brings some information about Y: here we assume that the larger S, the larger Y on average.
- Specifically, we assume that $s \mapsto \mathrm{E}[Y|S = s]$ is continuous and strictly increasing, that is, the response Y is positively regression dependent on the score.
- The more S is correlated to Y, the more information it contains.

- Idea: More informative scores lead to greater variability of the conditional expectation $\mathsf{E}[Y|S]$.
- See Ganuza and Penalva (2010), D (2010), Shaked, Sordo and Suarez-Llorens (2012).
- Consider two identically distributed scores S_1 and S_2, i.e.

$$F_{S_1} = F_{S_2}.$$

- Muliere and Petrone (1992), D (2010):

$(Y, S_1) \preceq_{(2,1)-\text{icx}} (Y, S_2)$

$$\begin{aligned} &\Rightarrow \quad \mathsf{E}[Y|S_1 > t] \leq \mathsf{E}[Y|S_2 > t] \text{ for all } t \\ &\Rightarrow \quad \mathsf{E}[Y|S_1] \preceq_{2-\text{icx}} \mathsf{E}[Y|S_2] \\ &\Rightarrow \quad \mathsf{E}\Big[\big(Y - \mathsf{E}[Y|S_2]\big)^2\Big] \leq \mathsf{E}\Big[\big(Y - \mathsf{E}[Y|S_1]\big)^2\Big]. \end{aligned}$$

Actuarial interpretation

- Assume that Y represents the annual claim amount for a policyholder in the portfolio.
- Then $\mathrm{E}[Y|S_i]$ represents the pure premium with risk classification based on $X_1, \ldots, X_p$, using score S_i.
- Notice that

$$\mathrm{E}[Y|S_1] \preceq_{2-\mathrm{icx}} \mathrm{E}[Y|S_2] \Rightarrow \mathrm{V}\Big[\mathrm{E}[Y|S_1]\Big] \leq \mathrm{V}\Big[\mathrm{E}[Y|S_2]\Big].$$

- Hence,

$$\mathrm{E}[Y|S_1 > t] \leq \mathrm{E}[Y|S_2 > t] \text{ for all } t$$

  ensures that there is more dispersion in the pure premiums with score S_2 compared to S_1.

Binary responses

- Consider a binary response $Y \in \{0, 1\}$.
- Gourieroux (1992) suggested to measure the performance of a score S using
$$t \mapsto \mathrm{P}[Y = 1|S > t].$$
- See also the book by Gourieroux and Jasiak (2011, Chapter 4) where $\mathrm{P}[Y = 1|S > t]$ defines the performance curve of the score S.
- Clearly,
$$\mathrm{P}[Y = 1|S > t] = \mathrm{E}[Y|S > t]$$
in this case so that we recover the conditional expectations used above.

Outline

Univariate stochastic dominance rules, D, Lefèvre and Shaked (1998)

Bivariate stochastic dominance rules, D, Lefèvre and Mesfioui (1999)

Expectation dependence and related concepts

Application 1: Asset allocation by prudent investors, D and Eeckhoudt (2016)

Application 2: Risk aversion with two risks, D and Mesfioui (2016)

Application 3: Model selection based on convex order, D and Trufin (2016)

Some references

- Caballe, J., Pomansky, A. (1996). Mixed risk aversion. Journal of Economic Theory 71, 485-513.
- Denuit, M. (2010). Positive dependence of signals. Journal of Applied Probability 47, 893-897.
- Denuit, M., Dhaene, J., Goovaerts, M.J., Kaas, R. (2005). Actuarial Theory for Dependent Risks: Measures, Orders and Models. Wiley, New York.
- Denuit, M., Eeckhoudt, L. (2010). Bivariate stochastic dominance and substitute risk (in)dependent utilities. Decision Analysis 7, 302-312.
- Denuit, M., Eeckhoudt, L., Menegatti, M. (2011). Correlated risks, bivariate utility and optimal choices. Economic Theory 46, 39-54.
- Denuit, M., Eeckhoudt, L., Rey, B. (2010). Some consequences of correlation aversion in decision science. Annals of Operations Research 176, 259-269.
- Denuit, M., Huang, R., Tzeng, L. (2015). Almost expectation and excess dependence notions. Theory and Decision 79, 375-401.
- Denuit, M., Lefèvre, Cl., Mesfioui, M. (1999). A class of bivariate stochastic orderings with applications in actuarial sciences. Insurance: Mathematics and Economics 24, 31-50.
- Denuit, M., Mesfioui, M. (2010). Generalized increasing convex and directionally convex orders. Journal of Applied Probability 47, 264-276.
- Denuit, M., Lefèvre, Cl., Scarsini, M. (2001). On s-convexity and risk aversion. Theory and Decision 50, 239-248.
- Denuit, M., Lefèvre, Cl., Shaked, M. (1998). The s-convex orders among real random variables, with applications. Mathematical Inequalities and Their Applications 1, 585-613.
- Denuit, M., Rey, B. (2010). Prudence, temperance, edginess, and risk apportionment as decreasing sensitivity to detrimental changes. Mathematical Social Sciences 60, 137-143.
- Denuit, M., Rey, B. (2013). Another look at risk apportionment. Journal of Mathematical Economics 49, 335-343.
- Eeckhoudt, L., Rey, B., Schlesinger, H. (2007). A good sign for multivariate risk taking. Management Science 53, 117-124.
- Eeckhoudt, L., Schlesinger, H. (2006). Putting risk in its proper place. American Economic Review 96, 280-289.
- Eeckhoudt, L., Schlesinger, H., Tsetlin, I. (2009). Apportioning risks via stochastic dominance. Journal of Economic Theory 144, 994-1003.

- Egozcue, M., Garcia, L.-F., Wong, W.-K., Zitikis, R. (2011). Grüss-type bounds for covariances and the notion of quadrant dependence in expectation. Central European Journal of Mathematics 9, 1288-1297.
- Finkelshtain, I., Kella, O., Scarsini, M. (1999). On risk aversion with two risks. Journal of Mathematical Economics 31, 239-250.
- Ganuza, J.-J., Penalva, J.S. (2010). Signal orderings based on dispersion and the supply of private information in auctions. Econometrica 78, 1007-1030.
- Gourieroux, C. (1992). Courbes de performance, de sélection et de discrimination. Annales d'Économie et de Statistique 28, 107-123.
- Gourieroux, C., Jasiak, J. (2011). The Econometrics of Individual Risk: Credit, Insurance, and Marketing. Princeton University Press.
- Guo, X., Li, J., Liu, D., Wang, J. (2016). Preserving the Rothschild-Stiglitz type of increasing risk with background risk. Insurance: Mathematics and Economics 70, 144-149.
- Hadar, J., Seo, T.K. (1988). Asset proportions in optimal portfolios. Review of Economic Studies 55, 459-468.
- Jokung, O. (2011). Risk apportionment via bivariate stochastic dominance. Journal of Mathematical Economics 47, 448-452.
- Kowalczyk, T., Pleszczynska, E. (1977). Monotonic dependence functions of bivariate distributions. Annals of Statistics 5, 1221-1227.
- Lehmann, E.L. (1966). Some concepts of dependence. Annals of Mathematical Statistics 37, 1137-1153.
- Li, J. (2011). The demand for a risky asset in the presence of a background risk. Journal of Economic Theory 146, 372-391.
- Li, J., Liu, D., Wang, J. (2016). Risk aversion with two risks: A theoretical extension. Journal of Mathematical Economics 63, 100-105.
- Li, J., Schlesinger, H., Yang, Z. (2015). Risk aversion and risk premiums with dependent risks. Proceedings of the 3rd World Risk and Insurance Economics Congress (WRIEC), Munich.
- Muliere, P., Petrone, S. (1992). Generalized Lorenz curve and monotone dependence orderings. Metron 50, 19-38.
- Shaked, M., Sordo, M. A., Suarez-Llorens, A. (2012). Global dependence stochastic orders. Methodology and Computing in Applied Probability 14, 617-648.
- Wright, R. (1987). Expectation dependence of random variables, with an application in portfolio theory. Theory and Decision 22, 111-124.